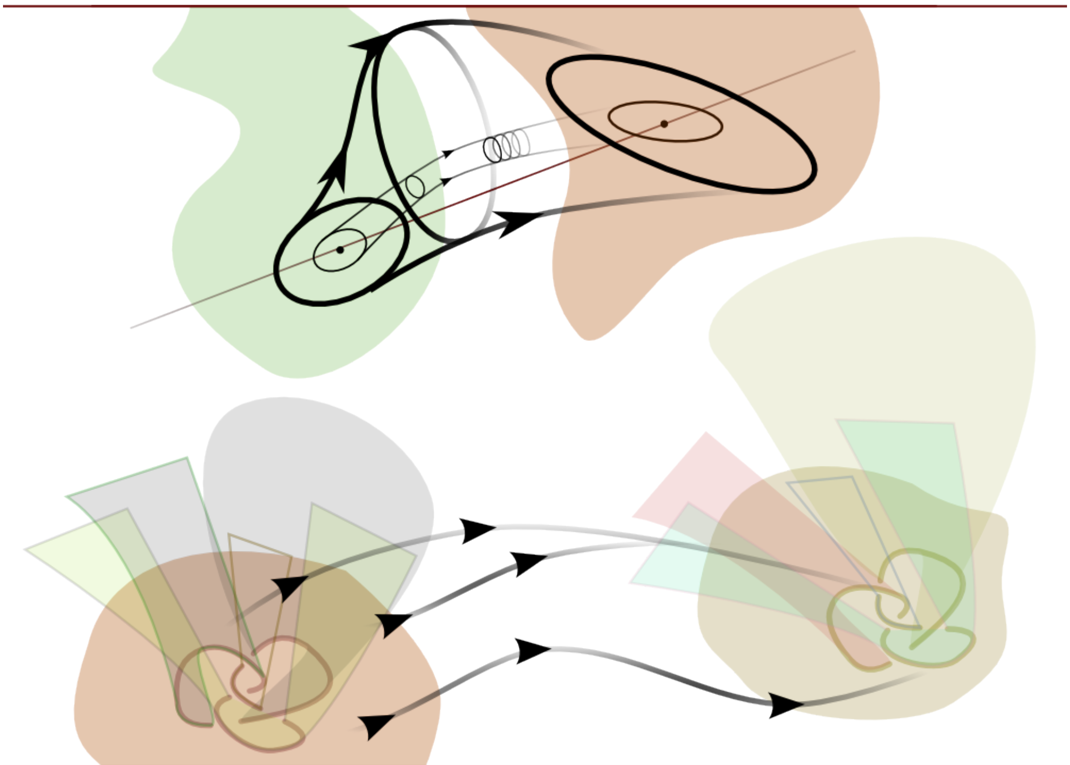
Utrecht University

Exotic path integrals and dualities

New views on quantum theory, gauge theories and knots

Master's thesis



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Abstract

In this thesis we review a recently discovered technique to give an alternative, equivalent expression for a given path integral, by finding a suitable alternative integration cycle. These alternative cycles are dubbed *exotic* and are found by exploiting basic properties of Morse theory in finite dimension and its generalizations to infinite dimensions. Combining this with supersymmetric localization and topological formulations of supersymmetry, this leads to a new duality between quantum mechanics and the topological A-model; this new point of view is related to the A-model view on quantization of classical systems. We discuss explicitly the subtle details in applying this technique to the harmonic oscillator. Another application of exotic cycles is to establish a new duality between Chern-Simons theory and topological $\mathcal{N} = 4$ super Yang-Mills. Embedding this system in type IIB superstring theory, using this duality and non-perturbative string dualities one can then give a conjectural gauge theory description of Khovanov homology. Furthermore, different facets and applications of exotic cycles will be discussed, as well as closely related current developments in mathematical physics, such as the role of S-duality and modularity in Chern-Simons theory.

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Introduction

Fundamental physics has had a long and fruitful interplay with pure mathematics, most notably, with the area of geometry. The canonical example is of course general relativity, which is an entirely geometric theory. However, quantum field theory provided a slight kink in this marriage, as the path integral of quantum field theory cannot be described rigorously using existing techniques, although being highly successful as a phenomenological model. Despite this drawback, more and more links between geometry and field theory have been established, provoking a renewed mathematical interest in the subject.

One of the starting points to understand the new connections between geometry and physics is the insight in 1982 that a complete physical description of Morse theory could be given in terms of supersymmetric quantum mechanics. This connected two, until then completely disparate, areas of science. Morse theory is concerned with the topological structure of smooth manifolds, whereas supersymmetry was primarily invented as an elegant extension of the Standard Model of elementary particles to combat the hierarchy problem.

Gauge theory, the mathematical framework of the Standard Model and its supersymmetric extensions has also been key in modern developments. Especially, supersymmetric gauge theory has been proven to calculate 4-manifold invariants, called Donaldson invariants, incorporating novel mathematical techniques, such as Floer theory. This provides another link between geometry and physics.

Another major stimulus for mathematical physics was the introduction of string theory, a mathematical model that provides a description of quantum gravity. Although its physical relevance remains partly to be seen, its mathematical virtues has already sparked a hausse of interest among mathematicians, since string theory has given rise to entirely new well-defined questions and areas of mathematics, moreover, has sometimes already provided answers where mathematicians had not. Especially, simplified versions of the physical string, the topological A and B-string that do not depend on the worldsheet metric, can be linked to counting holomorphic curves and the geometry of Calabi-Yau manifolds.

Finally, one of the most profound insights and examples is the connection that was made between 3-dimensional topological gauge theory and the computation of knot invariants. In 1989 it was shown that 3-dimensional Chern-Simons theory is completely non-perturbatively solvable through a dual description in terms of 2-dimensional conformal field theory. Moreover, one can prove in this description that Chern-Simons theory exactly computes knot polynomials. These objects are topological invariants associated to knots and links in 3 dimensions, which before were only known to mathematicians as purely algebraic constructions, from which topological invariance was not manifest. However, Chern-Simons theory gives an intrinsically topological description.

Hopefully, this is ample evidence that differential geometry and physics are tightly intertwined and deserve detailed scrutiny.

In this thesis the central theme is a recently introduced 'exotic duality' in topological gauge theory, which relates the path integrals of two completely different physical theories. Schematically, the exotic duality connects:

$$\boxed{d\text{-dimensional open supersymmetric } \sigma\text{-models} \longleftrightarrow d-1\text{-dimensional field theory.}}$$

The salient features of this duality are that it relates a supersymmetric theory to one that is not; moreover, it relates theories defined in different dimensions. Here 'open' refers to the σ-model with a boundary.

Roughly speaking, the exotic duality implies that the d-dimensional *bulk* theory reduces in the semi-classical limit to the dual theory, that lives on the *boundary* of the open σ-model. One of the intuitive explanations for the holographic nature of this duality comes from the appearance of Stokes' theorem in some parts of the construction: in nice situations, bulk behavior can be captured by data on the boundary.

To explain how this duality works, we need three main ingredients:

- Supersymmetric localization
- Topological field theory
- Morse theory

Supersymmetric localization is one of the fundamental reasons that supersymmetric field theory is quite elegant: supersymmetric path integrals simplify significantly as they can be evaluated by only considering certain fixed points of the supersymmetric theory.

Topological field theories are theories that are independent of the metric on the space-time on which they are defined. We shall discuss the two main classes of topological field theory: the first exploits a trick called twisting in order to define supersymmetry on curved space-times. For 2-dimensional σ-models, this results in the so-called A and B-model. The second class uses a metric-independent Lagrangian, which guarantees classical topological invariance; the canonical example is Chern-Simons theory.

Lastly, Morse theory studies the behavior of scalar functions on curved manifolds to describe their topology and differential structure. For instance, one can use the gradient flow of scalar functions to obtain bounds on the dimensions of the cohomology of a manifold. Most importantly, the nice behavior of scalar functions under gradient flow will be central in setting up the new duality.

While we will mostly use low and finite dimensional toy models to illustrate the power of these techniques, the most interesting applications require a formal generalization to infinite-dimensional manifolds to obtain the most interesting results. In order to do so, one needs to generalize the finite-dimensional Morse techniques to the infinite Floer theory techniques. However, this step is fraught with a lot of mathematical analysis, while not providing new relevant concepts. Therefore, in this thesis we shall mainly forego all the technical details that would be needed to set up Floer theory and instead argue informally why the finite-dimensional concepts generalize in a well-defined manner to the infinite-dimensional setting, by exploiting the elliptic nature of the relevant equations.

With this proviso, the first application that we describe is the duality between the

$$\boxed{\text{2-dimensional open A-model} \longleftrightarrow \text{1-dimensional quantum mechanics.}}$$

This duality will lead among others to a new view on quantization of classical theories. We shall illustrate in detail what the various subtleties are in applying this duality to the simple harmonic oscillator.

A-model quantization offers a new point of view on the inherently ambiguous process of 'quantization': there is no unique and completely systematic way to go from a given classical system to its quantum counterpart, even when the classical phase space is topologically trivial. When the phase space is topological non-trivial, things are even worse. These issues are relevant, for instance, in Chern-Simons theory which for compact G has a non-ambiguous complete quantization through its connection with 2-dimensional conformal field theory. This feature is key to solve it completely. However, when G is non-compact, the Chern-Simons phase space becomes highly non-trivial and non-compact, and its quantization is still mysterious. The latter situation would be physically interesting as it can be related to $2+1$-dimensional quantum gravity, mathematically one would like to understand knot invariants for non-compact G.

The second application of exotic integration cycles is to establish a bulk-boundary duality: we relate

$$\boxed{\text{4-dimensional twisted } \mathcal{N}=4 \text{ super Yang-Mills} \longleftrightarrow \text{3-dimensional Chern-Simons theory,}}$$

with the former in the bulk and the latter on its boundary. Together with its embedding in superstring theory, applying non-perturbative string dualities will lead to a new view on the Jones polynomial and on its categorification, known as Khovanov homology. Categorification refines knot invariants by assigning to a knot vector spaces, instead of numbers. This process generates a stronger knot invariant than the Jones polynomial, since categorification provides a richer algebraic structure. The essential point is that topological invariance of Khovanov homology is again not manifest: mathematicians only know an algebraic description of these knot invariants. The new exotic duality now proposes a gauge theoretic description of Khovanov homology, which does make topological invariance manifest.

We shall see that the exotic cycles establish a new link in a cascade of dual theories in consecutive dimensions, exemplifying the richness of topological gauge theory.

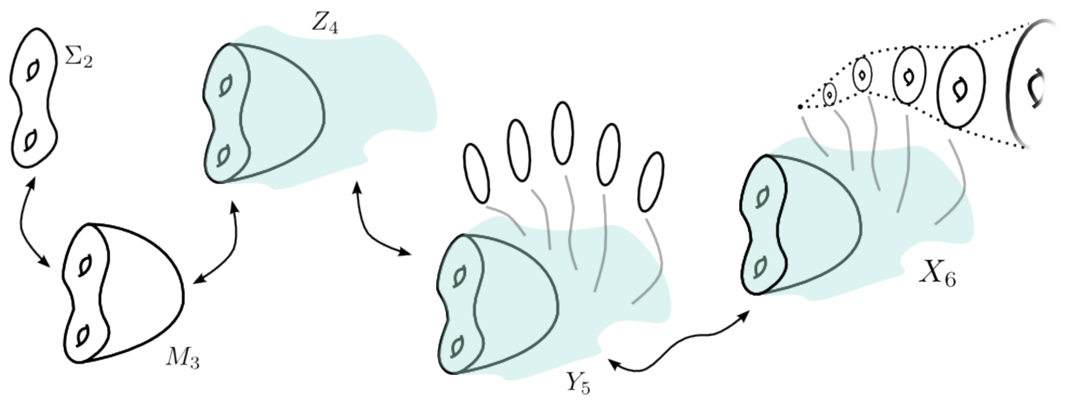

Figure 1: The cascade of gauge theories, each linked by duality relations. We have Wess-Zumino-Witten theory on Σ_2, Chern-Simons theory on $M_3, \partial M_3 = \Sigma_2$, twisted $\mathcal{N} = 4$ super Yang-Mills on $Z_4 = M_3 \times \mathbb{R}_-$, twisted super Yang-Mills on $Y_5 = M_3 \times \mathbb{R}_- \times S^1$, the $(0,2)$ CFT on $X_6 = M_3 \times D_2 \times S^1$, where D_2 is topologically $\mathbb{R}^2 = S^1 \times \mathbb{R}_-$, but inherits the circle fiber scaling of the Taub-NUT space $\mathcal{T}$ (see (10.2.1)). The first arrow is discussed in chapter 7, the last one in chapter 10 and the others in chapter 8.

We start in chapter 2 with a discussion of the basics of supersymmetric gauge theory on flat space-times and a key feature of supersymmetry: localization. In chapter 3 we will discuss how to extend supersymmetry to curved space-times by using the topological twist, which gives a topological field theory (TFT) that possess topological supersymmetry. We will discuss the key properties of TFTs and the so-called closed and open A-model as the main example.

In chapter 4 we discuss how to find alternative integration cycles for path integrals in low-dimensional QFT in chapter 4. In chapter 5 we expand these ideas to include gauge symmetry. Up to this point, we will mostly illustrate the techniques by applying them to toy models, such as 0-dimensional QFT.

Having then established the three main tools we will use, in chapter 6 we will use them to see how we can find a dual description of the path integral of quantum mechanics; this involves a generalization in which the relevant spaces, on which we apply Morse theory, will be infinite-dimensional. As an example, we will discuss in detail how this duality works for the simple harmonic oscillator.

We then continue with a discussion of Chern-Simons theory in chapter 7, before we show how the same techniques establish a duality between Chern-Simons theory on a boundary ∂V and twisted $\mathcal{N} = 4$ SYM in the bulk V in chapter 8. The motivation for this is to describe a conjecture for a gauge theory description of Khovanov homology, which is discussed in chapter 9. In chapter 10 we then end with a discussion of the implications of this new duality and current developments that are tightly intertwined with the two examples we discussed in chapter 6 and 8.

Since the topic of this thesis lies in the intersection of mathematics and physics, some (limited) mathematical background is needed. In an attempt to make this somewhat self-contained, some relevant material is briefly discussed in appendix A, with references to complete treatments: rigorous proofs can be found there, which we will forego here. Moreover, a short discussion of the relation of Morse theory and supersymmetric vacua can be found in appendix C. Some knowledge of differential geometry, basic algebraic topology, quantum field theory and superstring theory will be assumed.

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2

Supersymmetric gauge theory

The pillars of modern fundamental physics, general relativity and quantum field theory, can entirely be described in terms of differential geometry; especially, they can be entirely formulated in the language of fiber bundles. The structure of quantum field theory in particular is centered on the notion of *symmetry* (and the breaking thereof), which is encoded in the structure group of fiber bundles where physical fields live in. Such field theories are called gauge theories, we will take a look at one of the most important examples: Yang-Mills theory.

Another cornerstone of the link between geometry and physics is supersymmetry, which plays an essential role throughout. Here we will discuss supersymmetric gauge theory, viz. super Yang-Mills theory, and the main reason for the power of supersymmetry: the localization phenomenon. The latter will be used throughout the rest of our story.

2.1 Gauge theory

Let G be a compact Lie group and consider a principal G-bundle $E \longrightarrow M$ on an n-dimensional manifold M. We can think of M as playing the role of spacetime, but with possible non-trivial topology. We shall mainly work with manifolds with Euclidean signature. The structure group G is also called the *gauge group*. A connection 1-form A on the bundle is called the *gauge field*, and is a generalization of the familiar gauge potential of electromagnetism. Physical fields on M correspond to sections of the principal fiber E in a certain representation of G and the associated action of G on these sections and the connection are called *gauge transformations*. Physically, in gauge theories the gauge group G represents a *redundancy* in the system: there is an infinite number of equivalent descriptions of the same physical system.

The way G acts on fields depends on the representation the fields are in: if the physical fields sit in the fundamental representation, the group G acts by left multiplication: $\phi \overset{G}{\longmapsto} g\phi$. If the fields sit in the adjoint representation, the group G acts by conjugation: $\phi \overset{G}{\longmapsto} g\phi g^{-1}$. On fiber bundles one can introduce a *connection*: a derivation D that allows us to compare objects in the fiber bundle over different points in M. Such a connection can be written as an operation on forms $D = d + A$, where d is the de Rham differential and A is a $\mathfrak{g}$-valued 1-form on M. This last means that A is an element of $\Omega^1(M, \mathfrak{g})$ or more concretely, A can be written as $A = A_i^a T^a dx^i$, where $T^a \in \mathfrak{g}$ are generators of the group G.

Under a gauge transformation, the gauge field A transforms as $A \longmapsto gAg^{-1} - dgg^{-1}$. This is to ensure that the covariant derivative transforms naturally; if ψ is a section in the fundamental representation of the principal G-bundle, under a gauge transformation

$$D(g\psi) = d(g\psi) + (gAg^{-1}g\psi - dgg^{-1})(g\psi) = dg\psi + gd\psi + gA\psi - dg\psi = g(d\psi + A\psi) = gD\psi. \tag{2.1.1}$$

so $D\psi$ transforms as a section in the fundamental representation too. This requirement on the covariant derivative can be physically understood by the requirement that the kinetic energy term $(D^\mu\phi)^2$ remains invariant under gauge transformations.

The way D operates on sections of E depends on the representation they sit in. As the most common example, if ψ sits in the fundamental representation, then the 1-form A acts by left multiplication: $D\psi = d\psi + A\psi$. If ψ' sits in the adjoint representation, the covariant derivative acts on ψ' by the adjoint action on $\mathfrak{g}$: $D\psi' = d\psi' + \mathrm{ad}(A)\psi' = d\psi' + [A,\psi']$, where $[a,b] = a\wedge b - (-1)^{\deg a \deg b} b \wedge a$ is the graded bracket for Lie algebra valued forms. When we write D, we shall assume that this behavior is understood.

Non-abelian Yang-Mills theory and instantons

The example we consider is non-abelian gauge theory on $\mathbb{R}^4$, which describes a system of interacting gauge bosons. Here the gauge group G is a semisimple compact Lie group whose Lie algebra $\mathfrak{g}$ has antihermitian generators $\{T^a, a = 1\dots \dim G\}$ that obey the familiar relation $[T^a, T^b] = f^{ab}{}_c T^c$.* Here the $f^{ab}{}_c$ are the *structure constants* of the Lie algebra $\mathfrak{g}$ and are totally antisymmetric. The curvature 2-form of the connection $D_\mu = \partial_\mu + \lambda T^a A^a_\mu$ is in index notation

$$F_{\mu\nu} = [D_\mu, D_\nu] = \sum_{a=1}^{\dim G} F^a_{\mu\nu} T^a, \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \lambda \sum_{b,c=1}^{\dim G} f^{abc} A^b_\mu A^c_\nu. \tag{2.1.2}$$

where λ is the Yang-Mills coupling constant. One way to interpret F is that it measures to what degree the covariant derivative fails to be nilpotent. Under a local gauge transformation $A \longrightarrow gAg^{-1} - dgg^{-1}$, the curvature is conjugated $F \longmapsto gFg^{-1}$, which can be checked by a straightforward calculation.For brevity, we leave the sum over the index a implicit. We conclude from this that flatness $F = 0$ of a connection is preserved under gauge transformations. Therefore, we can divide the space of flat connections by gauge transformations, the resulting quotient is the moduli space of flat connections. We shall return to this subject in chapter 7. The action for this system is given by

$$\mathcal{S}_{YM} = -\frac{1}{2\lambda^2}\int_{\mathbb{R}^4} d^4x\, \mathrm{tr}\left(F_{\mu\nu}F^{\mu\nu}\right) = -\frac{1}{2\lambda^2}\int_{\mathbb{R}^4} \mathrm{tr}\left(F\wedge *F\right). \tag{2.1.3}$$

This is an intrinsically interacting system: the coupling constants of the cubic and the quartic interactions for the gauge boson are entirely determined by the structure constants of the Lie algebra. The equation of motion and Bianchi identity for A then become

$$D(*F) = 0, \qquad DF = 0. \tag{2.1.4}$$

The Bianchi identity is automatic; computing the exterior derivative of F:

$$\begin{aligned} dF &= d^2A + dA\wedge A - A\wedge dA = (F - A\wedge A)\wedge A - A\wedge(F - A\wedge A) \\ &= F\wedge A - A\wedge F = -(A\wedge F - (-1)^{1\cdot 2}F\wedge A) = -[A,F]. \end{aligned}$$

Here we have to use the graded Lie bracket for $\mathfrak{g}$-valued p-forms $[a,b] = a\wedge b - (-1)^{\deg a \deg b} b\wedge a$. The equations in (2.1.4) imply that a connection with self-dual or anti-self-dual curvature 2-form, satisfying $*F = \pm F$, automatically satisfies the equations of motion. Such classical solutions have *finite energy* and are known as *instantons (+ sign)* or *anti-instantons (- sign)*. This is our first encounter with such solutions and we will see that they play an important role throughout the rest of this thesis. To see that Yang-Mills instantons have finite energy, we add a *topological θ term*

$$\frac{i\theta}{8\pi^2}\,\mathrm{tr}\left(F\wedge F\right) \tag{2.1.5}$$

to the Lagrangian. This term is a total derivative, as will be shown in chapter 7 and hence only adds a constant to the action and does not change the equations of motion. It now follows from the Schwarz inequality $|\langle x,y\rangle|^2 \leq \langle x,x\rangle\langle y,y\rangle$ that

$$\int \mathrm{tr}\left(F\wedge F\right) = \left(\int \mathrm{tr}\left(F\wedge F\right)\int \mathrm{tr}\left(*F\wedge *F\right)\right)^{\frac{1}{2}} \geq \left|\int \mathrm{tr}\left(F\wedge *F\right)\right|, \tag{2.1.6}$$

where we used that $\langle \alpha,\beta\rangle = \int \mathrm{tr}\left(\alpha\wedge *\beta\right)$ is an inner product on 2-forms and $*\alpha\wedge *\beta = \alpha\wedge\beta$ for 2-forms on $\mathbb{R}^4$. We see that the energy of an instanton is bounded by its winding number, and the bound is

* There are two choices one can make in defining the generators T^a: either they are hermitian or anti-hermitian. In the previous notation, the covariant derivative was anti-hermitian, which is standard practice in mathematics. If the generators are hermitian, the covariant derivative should be written as $D = d - iA$ etc, which is more standard in physics. We shall mainly use the anti-hermitian convention, since it will get rid of irrelevant i's in formulas.

saturated only for instantons. The instanton equations $F^{\pm} = \frac{1}{2}(F \mp *F) = 0$ are called the self-duality $(+)$ or anti-self-duality $(-)$ equations. From a mathematical point of view, they are first-order elliptic (see for instance [1]), but non-linear, equations, which makes their analysis non-trivial. It is conventional to define the coupling constant

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{\lambda^2}, \tag{2.1.7}$$

so that the action can be written as

$$\mathcal{S}_{YM} = \frac{i\overline{\tau}}{8\pi}\int F^+ \wedge F^+ + \frac{i\tau}{8\pi}\int F^- \wedge F^-. \tag{2.1.8}$$

The important observation is that the θ-term computes topological information, since F is the curvature of a connection A on a principal G-bundle $E \longrightarrow M$. Through the definition of the total Chern class

$$c(E) = \det\left(1 + t\frac{iF}{2\pi}\right) = t^k c_k(E), \tag{2.1.9}$$

F is identified with the first Chern class $c_1(M) = \frac{iF}{2\pi}$. Each Chern class $c_k(E)$ sits in an integral cohomology class $H^k(M,\mathbb{Z})$: it follows that integrals over (powers of) $c_k(M)$ result in integer values. The geometric interpretation of the topological θ-term

$$\frac{\theta}{8\pi^2}\int_M \mathrm{tr}\, F \wedge F = -4\pi^2\frac{\theta}{8\pi^2}\int_M \mathrm{tr}\, c_1(E)^2 \in \mathbb{Z} \tag{2.1.10}$$

is that it computes the *winding number* of the associated gauge field A. The normalization of the trace is chosen in such a way that the winding numbers are integers. The winding number reflects the fact that not all gauge field configurations are homotopically equivalent to the trivial gauge field, it measures the degree in which such a non-trivial gauge field is 'twisted'.

2.2 Aspects of supersymmetry

Supersymmetry extends the standard Poincaré symmetry of $d+1$-dimensional space-time with fermionic symmetries, extending the symmetry group of space-time to the super-Poincaré symmetry group. By the *Coleman-Mandula theorem*, which states that the only conserved quantities in any viable quantum field theory with a mass gap are Lorentz scalars, extra fermionic symmetries are the only possible extension. Conventionally, the fermionic generators of the extra symmetries are denoted by Grassmann numbers $Q^A_\alpha, \overline{Q}^A_\beta = Q^\dagger\Gamma^0$, and the associated parameters are spinors $\epsilon^{A\alpha}$. Here Γ^μ are $d+1$-dimensional Γ-matrices that satisfy $\{\Gamma^\mu,\Gamma^\nu\} = 2\eta^{\mu\nu}$. Grouping these into fermionic raising and lowering operators $\Gamma^{\mu\pm}$, spinors sit in representations of this fermionic oscillator algebra. The α,β denote $d+1$-dimensional spinor indices and the $A = 1\ldots\mathcal{N}$ is an R-symmetry index, labeling the family of supersymmetry generators. R-symmetry is given by a $U(\mathcal{N})$ symmetry group that rotates the supercharges amongst themselves. We take the sign convention $(-,+,+,\ldots)$. The supersymmetry generators satisfy the following defining relation:

$$\left\{Q^A_\alpha, \overline{Q}^B_\beta\right\} = -2\delta^{AB}\Gamma^\mu_{\alpha\beta}P_\mu - 2iZ^{AB}\delta_{\alpha\beta}, \qquad [Q^A_\alpha, P^\mu] = 0. \tag{2.2.1}$$

Here P_μ is the generator of space-time translations, $\mu = 0,\ldots,d$ denotes space-time Lorentz indices and Z^{AB} is an antisymmetric matrix of *central charges*. We shall mostly set $Z^{AB} = 0$. Physical states of the theory sit in representations of the supersymmetry group, which are called *supermultiplets*. In each supermultiplet, there are equal numbers of bosonic and fermionic states on-shell and off-shell, where in the latter case auxiliary fields must be added to enforce the balance.

To study (2.2.1) in slightly more detail, we can consider how massless multiplets arise. First, we rewrite (2.2.1) as

$$\{Q^A_\alpha, (Q^\dagger)^B_\beta\} = -2\delta^{AB}(\Gamma^\mu\Gamma^0)_{\alpha\beta}P_\mu. \tag{2.2.2}$$

In the massless case, one can go to the frame where $P^\mu = (P, P, \ldots)$, upon which the right-hand side becomes the projection operator $2\delta^{AB}\left(1+\Gamma^0\Gamma^1\right)_{\alpha\beta}$, which vanishes on half of all particle states.[§] We see that half of the Qs vanish, then the remaining Q can be split into nilpotent raising and lowering operators. To go from spin 2 to spin -2, one needs 8 lowering operators; there is no consistent way to write down a field theory for particles with spin larger than 2. Hence, the largest number of real supercharges is 32. For instance, in four dimensions, the supersymmetry generators are Weyl spinors with four real components. Hence, in four dimensions $\mathcal{N}$ can be at most 8. Similar considerations hold for massive multiplets: there one finds 2 copies of the fermionic oscillator algebra, which doubles the number of states in the supermultiplets. The details of these constructions can be found for instance in the appendix of [2].

Since the supersymmetry generators Q are fermionic, we need fermions ϵ parametrizing supersymmetry (as $\exp \epsilon Q$ has to be bosonic). We shall only consider globally constant supersymmetry parameters ϵ, which gives so-called rigid supersymmetry. Non-rigid supersymmetry parametrized by local ϵ would lead to supergravity, as the gauge field for local supersymmetry would be a spin 3/2 particle, a gravitino. To preserve supersymmetry in that case, one is forced to add spin-2 particles, the gravitons. Since spinors always exist locally (under the assumption that the space-time is spin), but may fail to be defined globally, defining supersymmetry on a curved manifold generically is problematic. We will return to this issue in the next chapter.

Super Yang-Mills theory

Here we discuss maximally supersymmetric super Yang-Mills (SYM) gauge theory, which in four dimensions is precisely $\mathcal{N}=4$ SYM. The adjective maximally supersymmetric stems from the fact that in 4 dimensions, Weyl spinors have 4 complex components, and so for $\mathcal{N}=4$ supersymmetry generators, on-shell there are 16 real supersymmetry generators in total. This means that the supermultiplets are as large as possible without containing spin-2 particles. The reference for this section is [3].

The field content of $\mathcal{N}=4$ SYM can be most easily determined from dimensional reduction of $\mathcal{N}=1$ SYM in 10 dimensions (although this is not the only possibility, one could also try to determine it by brute-force). The action of 10-dimensional $\mathcal{N}=1$ super Yang-Mills is given by

$$I_{10} = \frac{1}{g_{10}^2}\int d^{10}x \,\mathrm{tr}\left(\frac{1}{2}F_{IJ}F^{IJ} - i\bar{\lambda}\Gamma^I D_I\lambda\right) \tag{2.2.3}$$

where $I, J = 0, \ldots, 9$ are 10d indices and λ is a 10d Majorana-Weyl, chiral real spinor*. F_{IJ} is the curvature of the 10-dimensional gauge field A_I. g_{10} is the 10-dimensional Yang-Mills coupling constant. The covariant derivative acts on spinors as

$$D_\mu\lambda^a = \partial_\mu\lambda^a + g_{10}f_{abc}A_\mu^b\lambda^c. \tag{2.2.4}$$

The generator of supersymmetry is a constant chiral spinor $\epsilon \in S_+$, that obeys $\overline{\Gamma}\epsilon = \epsilon$. Here, S_+ is the spinor bundle of positive chirality. The fields sit in one supermultiplet (A_I, λ). The supersymmetry transformations associated to ϵ for any field Φ are denoted as $\delta\Phi = \left[\sum_{a=1}^{16}\epsilon^a Q_a, \Phi\right\}$ (here $[.\}$ denotes an anti-commutator if Φ is fermionic, or commutator if it is bosonic). In this case, they read

$$\delta A_I = i\bar{\epsilon}\Gamma_I\lambda, \qquad \delta\lambda = \frac{1}{2}\Gamma^{IJ}F_{IJ}\epsilon, \qquad \delta\bar{\lambda} = -\frac{1}{2}\bar{\epsilon}\Gamma^{IJ}F_{IJ}, \tag{2.2.5}$$

where we define $\Gamma^{IJ} = \Gamma^I\Gamma^J$ (not the antisymmetric product!). Under these supersymmetry transformations, the action (2.2.3) is invariant up to total derivatives: this follows from properties of Fierz identities

[§] This follows from the oscillator algebra: states can represented by their eigenvalues $s_\mu = \pm\frac{1}{2}$ of the operators $S_\mu = \Gamma^{\mu+}\Gamma^{\mu-} - \frac{1}{2}$. In particular $\Gamma^{0\pm} = \frac{1}{2}\left(\pm\Gamma^0 + \Gamma^1\right)$. Then $\Gamma^0\Gamma^1 = 2S_0$.

*Note that this theory is chiral and therefore suffers from the axial anomaly, which vanishes only for SYM with gauge groups $G = SO(32), E_8\times E_8$ coupled to SUGRA: this is the main reason for the consistency of type I and heterotic superstrings.

in 10 dimensions [3]. The supercurrent associated to supersymmetry of the theory, is the Noether current[†]

$$J^I = \frac{1}{2}\,\mathrm{tr}\left(\Gamma^{JK}F_{JK}\Gamma^I\lambda\right) \tag{2.2.6}$$

Note that the trace here is with respect to the gauge group G. The computations that leads to these results can be found in volume 1 of [4].

Any Dirac spinor in d dimensions has $2^{[\frac{d}{2}]}$. complex degrees of freedom. Imposing a Weyl (chirality) or Majorana (realness) constraint each cuts down the number of real degrees of freedom by half. Putting the theory *on-shell* eliminates yet another half of the degrees of freedom. Hence λ has $2\cdot 2^{\frac{10}{2}}/8 = 32/4 = 8$ on-shell real degrees of freedom. Furthermore, there is a gauge field A_I with 8 physical polarizations whose field strength is F_{IJ}. Adding, we see that there are 16 on-shell real degrees of freedom in this theory.

We now describe the result of dimensional reduction of this theory to 4d, by declaring fields to be only dependent on $X^I, I = 0,\dots 3$, which means we break the 10-dimensional Lorentz group $SO(1,9) \to SO(1,3)\times SO(6)$. The residual $SO(6)$ rotates the internal coordinates, and becomes the R-symmetry group of the theory: R-symmetry is an internal symmetry. This means that we can set all derivatives in the $4\dots 9$-directions to zero. For the fermionic fields, we have to decompose the 10-dimensional Γ-matrices:

$$\Gamma_\mu \to \gamma_\mu \otimes \mathbb{I}_8, \qquad \Gamma_y \cong \gamma_{ij} = \gamma_5 \otimes \begin{pmatrix} 0 & \rho_{ij} \\ \rho^{ij} & 0 \end{pmatrix} \tag{2.2.7}$$

where the $\gamma_\mu, \mu = 0,1,2,3$ are the 4-dimensional γ-matrices and the 4×4 ρ-matrices are defined by

$$(\rho_{ij})_{kl} = \epsilon_{ijkl}, \qquad (\rho^{ij})_{kl} = \frac{1}{2}\epsilon^{ijmn}\epsilon_{mnkl}. \tag{2.2.8}$$

The chiral matrix and charge conjugation matrices decompose as

$$\Gamma_{chir} = \gamma_5 \otimes \mathbb{I}_8, \qquad \mathcal{C}_{10} = \mathcal{C}\otimes \begin{pmatrix} 0 & \mathbb{I}_4 \\ \mathbb{I}_4 & 0 \end{pmatrix}. \tag{2.2.9}$$

Under this decomposition, the 10-dimensional spinor splits up as

$$\lambda = \begin{pmatrix} L\chi^i \\ R\tilde{\chi}_i \end{pmatrix}, \qquad L = \frac{\mathbb{I}+\gamma_5}{2}, \qquad R = \frac{\mathbb{I}-\gamma_5}{2} \tag{2.2.10}$$

where $\chi^i, \tilde{\chi}_i, i = 1,2,3,4$ are chiral Weyl spinors and χ^i satisfies the Majorana condition $\tilde{\chi}_i = \mathcal{C}\overline{\chi}^{i,t}$. This ensures that λ is a 10-dimensional Majorana-Weyl spinor. One then gets that the field content of 4d $\mathcal{N}=4$ SYM is:

- 1 gauge field A^μ: a vector with 2 physical polarizations ~ 2 real degrees of freedom
- 6 scalars $\phi_i, i = 1,\dots 6$ that can be combined into a spin-0 antisymmetric 2-form $\varphi_{ij} \sim 6$ real degrees of freedom.
  The two form is defined by $\varphi^*_{ij} = \frac{1}{2}\epsilon^{ijkl}\varphi_{kl} = \varphi^{ij}$, $\varphi_{i4} = \frac{1}{\sqrt{2}}\left(\phi_{i+3} + i\phi_{i+6}\right)$.
- 4 spin-1/2 chiral Weyl spinors: 2 left $\chi^{i\alpha}$, 2 right $\overline{\chi}^{i\dot{\alpha}} \sim 4\times 2$ *on-shell* real degrees of freedom

where the i,j are 4d space-time indices. Adding all contributions, we have 16 real degrees of freedom, exactly the same as in $\mathcal{N}=1$ $d=10$ SYM. By simply inserting the dimensionally reduced versions of the

[†]We recall the standard result that if the Lagrangian changes under a variation δX by $\delta L = \partial_\mu K^\mu$, then the Noether current is given by $J^\mu = \left(\frac{\delta L}{\delta(\partial_\mu X)}\delta X - K_\mu\right)$.

10-dimensional fields into the 10-dimensional Lagrangian, one may check that the action then becomes (with rescaling):

$$\int d^4x \,\mathrm{tr}\left(\frac{1}{2}F^{\mu\nu}F_{\mu\nu} - D_\mu \varphi_{ij} D^\mu \varphi^{ij} - i\overline{\chi}_i \sigma^\mu D_\mu \chi^i \right. \tag{2.2.11}$$

$$\left. -4g_4^2[\varphi_{ij}, \varphi_{kl}][\varphi^{ij}, \varphi^{kl}] + g_4 \chi^i [\chi^j, \varphi_{ij}] + g_4 \overline{\chi}_i [\overline{\chi}_j, \varphi^{ij}]\right) \tag{2.2.12}$$

Here g_4 is the 10-dimensional Yang-Mills coupling constant. The supersymmetry variations then reduce to

$$\begin{aligned}
\delta A^\mu &= -i\chi^{j\alpha}\sigma^\mu_{\alpha\dot{\alpha}}\overline{\epsilon}^{\dot{\alpha}}_j - i\epsilon^{j\alpha}\sigma^\mu_{\alpha\dot{\alpha}}\overline{\chi}^{\dot{\alpha}}_j \\
\delta\chi^i_\alpha &= \frac{1}{2}F_{\mu\nu}\sigma^{\mu\nu}{}_\alpha{}^\beta \epsilon^i_\beta + 4i(\not{D}_{\alpha\dot{\alpha}}\varphi^{ij})\overline{\epsilon}^{\dot{\alpha}}_j - 8g[\overline{\varphi}_{jk}, \varphi^{ki}]\epsilon^j_\alpha \\
\delta\varphi^{ij} &= \frac{1}{2}\left(\chi^{i\alpha}\epsilon^j_\alpha - \chi^{j\alpha}\epsilon^i_\alpha\right) + \frac{1}{2}\epsilon^{ijkl}\overline{\epsilon}_{k\dot{\alpha}}\overline{\chi}^{\dot{\alpha}}_l
\end{aligned}$$

Recall that the R-symmetry of the $\mathcal{N} = 4$ theory is $SO(6)_R \cong SU(4)_R$, which rotates the 4 4-dimensional supercharges Q^A_α. R-symmetry rotates the fermions, which sit in a spinor **4** representation of Spin$(6)_R \cong SU(4)_R$ and the 6 scalars ϕ_i sit in a vector $\mathbf{6}_v$ of $SO(6)_R$.*

Here we can also add a supersymmetric topological θ-term, just as in the non-supersymmetric case. Then the bosonic part of the 4d action, can be written as

$$I_4 = \frac{1}{g_{10}^2}\int d^4x \,\mathrm{tr}\left(\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + D_\mu\phi_i D^\mu \phi_i + \frac{1}{2}\sum_{i,j=1}^{6}\left[\phi_i, \phi_j\right]^2\right) - \frac{\theta}{8\pi^2}\int \mathrm{tr}\,(F\wedge F)\,. \tag{2.2.13}$$

This will be a crucial addition for us, as we will encounter this term throughout the text. It also will be discussed in more detail in chapter 8, where we will study boundary conditions of super Yang-Mills with a topological θ-term.

One very important property of the $\mathcal{N} = 4$ theory is the strong result that its β-function vanishes for all values of the couplings, implying that the theory is conformal at all energy scales. This is a rather non-trivial statement, but has been proven perturbatively by Mandelstam and non-perturbatively by Seiberg [5]. Hence the full symmetry group of $\mathcal{N} = 4$ super Yang-Mills is actually the superconformal group.

2.3 Localization and supersymmetry

The power of supersymmetric theories comes from the underlying principles of localization and deformation invariance. These phenomena permeate all supersymmetric discussions and account for the elegance of supersymmetric theories. Let us illustrate this with a toy example.

Consider a 0-dimensional supersymmetric QFT, where the base manifold is 0-dimensional (a point p) and the target space is $\mathbb{R}$. We define a supersymmetric QFT which has a bosonic scalar field X and two real Grassmann variables ψ_1, ψ_2. Then the most general Lagrangian or action (in 0 dimensions, there is no distinction) is

$$S(X, \psi_1, \psi_2) = S_0(X) - \psi_1\psi_2 S_1(X). \tag{2.3.1}$$

The Euclidean path integral reduces to an ordinary integral over the 'variables' X, ψ_1, ψ_2:

$$Z = \int dX d\psi_1 d\psi_2 \exp\left(-S(X, \psi_1, \psi_2)\right) = \int dX S_1(X) \exp\left(-S_0(X)\right). \tag{2.3.2}$$

* The 10-dimensional gauge field A_I transforms as $A_I \to \Lambda_I^J A_J(\Lambda^{-1}x)$ under 10-dimensional Lorentz transformations. When we break $SO(1,9) \to SO(1,3)\times SO(6)$, an 'Lorentz' transformation takes the form $\begin{pmatrix}\Lambda & 0\\ 0 & R\end{pmatrix}$, and it is straightforward to see that $SO(6)$ rotates only $A_I, I = 4, \ldots 9$ and Λ rotates only $A_I, I = 0, \ldots 3$.

with the convention $\int d\psi_1 d\psi_2 \psi_1 \psi_2 = 1$. Let us make a supersymmetric choice for the action

$$S(X, \psi_1, \psi_2) = \frac{1}{2}(\partial h)^2 - \psi_1 \psi_2 \partial^2 h \tag{2.3.3}$$

where $h = h(X)$ is some function. Then this action is invariant under the transformations

$$\delta X = \epsilon \psi_1 + \epsilon \psi_2, \qquad \delta \psi_1 = \epsilon \partial h, \qquad \delta \psi_2 = -\epsilon \partial h. \tag{2.3.4}$$

where ϵ is a fermionic parameter. Indeed we have

$$\begin{aligned} \delta(\partial h)^2 &= 2\partial h \delta \partial h = 2\partial h \partial(\delta h) = 2\partial h \partial(\partial h \delta X) = 2\partial h \partial(\partial h(\epsilon(\psi_1 + \psi_2))) \\ &= 2(\epsilon(\psi_1 + \psi_2))\partial h \partial^2 h \\ \delta(\partial^2 h) &= \partial^2(\delta h) = \partial^3 h \delta X = \epsilon(\psi_1 + \psi_2)\partial^3 h \end{aligned}$$

from which we obtain

$$\begin{aligned} \delta S &= \epsilon(\psi_1 + \psi_2)\partial h \partial^2 h - \epsilon \partial h \psi_2 \partial^2 h + \psi_1 \epsilon \partial h \partial^2 h - \psi_1 \psi_2 \epsilon(\psi_1 + \psi_2)\partial^3 h \\ &= \epsilon(\psi_1 + \psi_2)\partial h \partial^2 h - \epsilon \partial h \psi_2 \partial^2 h - \epsilon \psi_1 \partial h \partial^2 h - \psi_1 \psi_2 \epsilon(\psi_1 + \psi_2)\partial^3 h = 0. \end{aligned}$$

We can now use the fermionic symmetry to eliminate one fermionic field, say ψ_2 and trade the $d\psi_1$ integration for a trivial ϵ integration, which amounts to a fermionic version of the Fadeev-Popov trick. Consider a bosonic analogue: suppose we have an integral $I = \int_{\mathbb{R}^2} dx dy g(x, y)$ and we knew that g was rotation invariant. Instead factoring out the angular θ-integration that contributes a factor of 2π and performing the radial integral, we instead can employ the Fadeev-Popov trick here. Using the delta function identity

$$\int dx \delta[f(x)] = \sum_{\text{roots of } f} 1/f'(x_i) \tag{2.3.5}$$

and rotated coordinates $x' = x\cos\theta - y\sin\theta, y' = y\cos\theta + x\sin\theta$, we find the Fadeev-Popov determinant

$$\Delta(x', y')^{-1} = \int d\theta \delta[f(x', y')] = \int d\theta \delta[y'] = \int d\theta \delta[y\cos\theta + x\sin\theta] = \frac{1}{x'} \tag{2.3.6}$$

where the gauge-fixing condition is $f(x, y) = y$. Inserting 1 into our original integral, we have

$$\begin{aligned} I &= \int_{\mathbb{R}^2} dx dy g(x, y)\Delta(x, y) \int d\theta \delta(y') = \int d\theta dx' dy' g(x', y')\Delta(x', y')\delta[y'] \\ &= \int d\theta \int dx' dy' g(x', y')\Delta(x', y')\delta[y'] = \int d\theta \int dx' x' g(x', 0) = 2\pi \int dx' x' g(x', 0), \end{aligned}$$

which is what we expect. But now we can see what goes wrong in the fermionic case: by analogy we would like to write down an expression like

$$Z = \int d\epsilon \int dX' d\psi_2' \exp\left(-S(X', 0, \psi_2')\right) \Delta \tag{2.3.7}$$

but this expression is 0, since $\int d\epsilon \,\{\text{'independent of } \epsilon\text{'}\} = 0$. However, the partition function should not vanish; the resolution of this paradox comes from the Fadeev-Popov determinant Δ, which is

$$\Delta(X, \psi_1, \psi_2)^{-1} = \int d\epsilon \delta[f(X', \psi_1', \psi_2')] = \int d\epsilon \delta[\psi_1'] = \int d\epsilon \delta[\psi_1 + \epsilon \partial h] = \int d\epsilon(\psi_1 + \epsilon \partial h) = \partial h,$$

where the gauge-fixing function f is such that ψ_1' is set to zero. The partition function therefore is

$$Z = \int d\epsilon \int dX' d\psi_2' \exp\left(-S(X', 0, \psi_2')\right) \frac{1}{\partial h}. \tag{2.3.8}$$

We see that the only contributions come from critical points of h: the theory *localizes* on fixed points of the fermionic Q-variations where the Fadeev-Popov trick breaks down. The simplest consequence is that the partition function Z is not zero, only critical points of h or fixed points of the supersymmetry variations, contribute to the path integral. This phenomenon persists in all supersymmetric theories, whenever the action is invariant under a fermionic symmetry transformation. The same reasoning as above then applies. This technique will be applied throughout the rest of this text.

Another point of view on the localization phenomenon is *deformation invariance*. Under $h \longmapsto h + \rho$ the action changes by $\delta_\rho S = \partial h \partial \rho - \partial^2 \rho \psi_1 \psi_2$, and we have $\delta_\epsilon(\partial \rho \psi_1) = \partial^2 \rho \delta X \psi_1 + \partial \rho \delta \psi_1 = \epsilon \left(\partial \rho \partial h - \partial^2 \rho \psi_1 \psi_2 \right)$. So $\delta_\rho S = \delta_\epsilon(\partial \rho \psi_1)$ is Q-exact. But this implies that the action is invariant under an infinitesimal rescaling of ρ! This is true as long as ρ is small at infinity in field space, otherwise $\langle \delta g \rangle = \int \delta g e^{-S} = \int \delta(g e^{-S}) \neq 0$ due to a boundary contribution. If h is a polynomial of order n, then ρ can also be of degree n, as long as its leading term is smaller than h. In particular, we can choose ρ such that we rescale $h \longmapsto th$. After rescaling the partition function becomes

$$Z = \int dX d\psi_1 d\psi_2 \exp\left(-(\partial h)^2/2 + \psi_1 \psi_2 \partial^2 h \right) = \int dX \exp\left(-t^2 (\partial h)^2/2 \right) t \partial^2 h. \tag{2.3.9}$$

Since Z is insensitive to rescaling of h, we can take $t \longrightarrow \infty$, and the only contributions come from critical points of h. But by identifying $t^2 = 1/\hbar$, this is just the semi-classical approximation: we see that the semi-classical approximation is exact. This continues to hold for all supersymmetric QFTs, which simplifies the theory enormously.

3

Topological field theory

Topological field theories (TFTs) are toy models of full quantum field theories that generically only detect global topological properties of the spacetime M they are defined on. The reason for this is that TQFTs are independent of the metric on M. TQFTs come in two types: *Witten-type* and *Schwarz-type*.

In this chapter we discuss those of Witten-type: they are constructed by *twisting*, a procedure in which the internal symmetries of a (metric dependent) theory are combined to obtain an enhanced BRST-like symmetry, which ensures that the theory becomes metric-independent. We will discuss those of Schwarz-type, the important example of which is Chern-Simons theory, in chapter 7.

The topological model we will study here is the A-twisted σ-model with open and closed worldsheet. In the open A-model, we discuss the concept of topological branes and their characterization. The motivation to do so is that the open A-model can be related to quantum mechanics, as we will show in chapter 6.

3.1 Cohomological field theory

As we saw at the end of chapter 2.3, localization and deformation invariance were the source of the power of supersymmetric field theories. This behavior also occurs in a large class of topological theories, namely those of cohomological type. These are defined by the existence of a symmetry Q which satisfies:

- Q is nilpotent. Denoting the infinitesimal transformations generated by Q by $i\epsilon\delta\mathcal{O} = \{Q, \mathcal{O}\}$, we have $\delta^2 = 0$ or $Q^2 = 0$.
- The ground state is annihilated by Q: $Q|0\rangle = 0$.
- Observables obey $\{Q, \mathcal{O}\} = 0$.

Here it is understood that $\{Q, .\}$ is the anticommutator acting on fermions and a commutator acting on bosons. Because the topological symmetry generator Q is nilpotent, it is referred to, due to historical reasons, as a BRST symmetry. The structure of the ring of observables is by the nilpotency of Q, entirely analogous to that of cohomology: observables sit in the cohomology of Q: any observables $\mathcal{O}$ obeys $\{Q, \mathcal{O}\} = 0$ and we identify $\mathcal{O} \sim \mathcal{O} + \{Q, X\}$ where X is an arbitrary observable.

Furthermore, we require that the action S is Q-exact, $S = \{Q, V\}$, for some V, which is often called the *gauge fermion*. This immediately implies that the stress energy tensor $T_{\mu\nu}$ is *Q-exact*, since

$$T_{\mu\nu} = \frac{\delta I}{\delta g^{\mu\nu}} = \{Q, \frac{\delta V}{\delta g^{\mu\nu}}\} = \{Q, b_{\mu\nu}\}, \tag{3.1.1}$$

where $g_{\mu\nu}$ is an appropriate metric. These theories are referred to as *cohomological topological field theories*. The topological nature of the theory follows from considering, for instance, the partition function of the theory, which is given by

$$Z = \int \mathcal{D}\phi \exp(-S[\phi]), \tag{3.1.2}$$

where ϕ represents the field content of the theory. Since $T_{\mu\nu} = \delta S[\phi]/\delta g^{\mu\nu}$, we find that

$$\frac{\delta Z}{\delta g^{\mu\nu}} = \int \mathcal{D}\phi - \frac{\delta S[\phi]}{\delta g^{\mu\nu}} \exp\left(-S[\phi]\right) = -\langle\{Q, b_{\mu\nu}\}\rangle, \tag{3.1.3}$$

where the bracket represents a vacuum expectation value. Since the ground state is annihilated by Q and $Q^\dagger$, this expectation value must vanish and so Z must, at least formally, be metric-independent. In fact, the expectation value of a combination of $\{Q, V\}$ and other observables vanishes for any V, since the vacuum must be invariant under the symmetry generated by Q.[‡] Explicitly, we should have that for any set of observables

$$\begin{aligned}\langle 0|\mathcal{O}_1 \ldots \mathcal{O}_i\{Q,V\}\mathcal{O}_{i+1}\ldots\mathcal{O}_n 0|\rangle &= \langle 0|\mathcal{O}_1\ldots\mathcal{O}_i\,(QV \pm VQ)\,\mathcal{O}_{i+1}\ldots\mathcal{O}_n|0\rangle \\ &= \pm\langle 0|Q\mathcal{O}_1\ldots\mathcal{O}_i\mathcal{O}_{i+1}\ldots\mathcal{O}_n|0\rangle \\ &\quad +\epsilon\langle 0|\mathcal{O}_1\ldots\mathcal{O}_i\mathcal{O}_{i+1}\ldots\mathcal{O}_n Q|0\rangle \\ &= 0\end{aligned}$$

in the process there appears an irrelevant sign $\epsilon = \pm 1$. We were allowed to shift Q to the far left and right by Q-closedness of observables $\{Q, \mathcal{O}_i\} = Q\mathcal{O}_i \pm \mathcal{O}_i Q = 0$.

Now the semiclassical approximation is exact by deformation invariance: inserting a parameter $\hbar$ into the path integral, we have

$$Z = \int \mathcal{D}\phi \exp\left(-\frac{1}{\hbar}S[\phi]\right) \Rightarrow \frac{\delta Z}{\delta\hbar^{-1}} = -\langle\{Q,V\}\rangle = 0. \tag{3.1.4}$$

Hence, Z is independent of $\hbar$ and we can calculate Z *exactly*, in the limit that $\hbar \longrightarrow 0$, which is exactly the semiclassical approximation. From this, we learn that the theory *localizes* on field configurations for which $I = \{Q, V\} = 0$.

3.2 Supersymmetry on curved manifolds: the supersymmetric twist

Supersymmetry is parametrized by a supersymmetry spinor $\epsilon^{A\alpha}$. On flat space $\mathbb{R}^n$, there are no issues in defining supersymmetry globally, since all fiber bundles on flat $\mathbb{R}^n$ are trivial. Infinitesimal supersymmetry transformations are expressions of the generic form $\delta\Phi^i = \epsilon Q\Phi^i$ (i is a target space index): these should be defined everywhere on M in order to show that the action is supersymmetric. If ϵ is covariantly constant, $D\epsilon = 0$, we can pull ϵ outside covariant derivatives and conclude that for any variation ϵ the action is invariant.

For σ-models with flat worldsheet, one usually singles out a time direction: we set $\Sigma = Y \times \mathbb{R}$ where $Y = S^1$ in the compact case. This means that for global worldsheet supersymmetry, we only need a covariantly constant spinor of Y, which on a circle would just have to be constant. However, on a general curved worldsheet, we cannot single out such a time direction and generically there are no global sections of the spinor bundle on a curved worldsheet. As a bosonic analogy, the *hairy ball theorem* shows that there is no global non-vanishing vector field on S^2.[*] Even worse, if a covariantly constant object vanishes somewhere, it vanishes everywhere. This is the main obstruction to defining supersymmetry globally on a general curved manifold.

So to obtain topological quantum field theories on a curved manifold, we need a trick to construct a globally defined supersymmetry generator Q. The key observation is that *scalar* objects are always globally defined: for instance, the bundle of smooth functions on M is always trivial. Therefore, if we can change (part of) the supersymmetry spinor Q to be a scalar, we will have a globally defined (partial) supersymmetry generator. This procedure is called *twisting*.

[‡]This follows from $\langle\Psi|H|\Psi\rangle = \langle\Psi|\{Q,Q^\dagger\}|\Psi\rangle = ||Q|0\rangle||^2 + ||Q^\dagger|0\rangle||^2 \geq 0$ for any state $|\Psi\rangle$. So for a supersymmetric vacuum, we need $Q|0\rangle = Q^\dagger|0\rangle = 0$

[*] *The hairy ball theorem on* $\mathbb{C}P^1$. Choosing local holomorphic coordinates $(z,\bar{z})$ on the Riemann sphere $\mathbb{C}P^1$, and considering the Kähler metric $h = dzd\bar{z}/(1+|z|^2)^2$ shows that the curvature 2-form of the tangent bundle $T\mathbb{C}P^1$ is given by $\Omega = 2dz\wedge d\bar{z}/(1+|z|^2)^2$. Since $c_1(T\mathbb{C}P^1) = \frac{i}{2\pi}\Omega$, we compute $\int c_1(T\mathbb{C}P^1) = \frac{i}{\pi}\int\frac{dz\wedge d\bar{z}}{(1+|z|^2)^2} = 2$. Since any trivial bundle $V \longrightarrow M$ has $\int c_1(V) = 0$, we see that $T\mathbb{C}P^1$ is not trivial, hence does not possess a global section.

The $\mathcal{N} = (2,2)$ 2d σ-model

We pick a flat Riemann surface Σ, the worldsheet, and a target space manifold M. Then the 2d σ-model describes bosonic embeddings $\Phi : \Sigma \longrightarrow M$. Picking local coordinates x^i on M, Φ is given in local coordinates by $\phi^i = x^i \circ \Phi, i = 1, \ldots, \dim M$. Locally we can always find a flat Euclidean metric whose components are given by $g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}, g_{zz} = g_{\bar{z}\bar{z}} = 0$. The action is defined such that minimization of the action corresponds to minimization of the area of the worldsheet. Including $\mathcal{N} = (1,1)$ supersymmetry on the worldsheet gives the action

$$S = \int_\Sigma d^2z \left(\frac{1}{2} g_{ij} \partial_z \phi^i \partial_{\bar{z}} \phi^j + \frac{i}{2} g_{ij} \psi_-^i D_z \psi_-^j + \frac{i}{2} g_{ij} \psi_+^i D_{\bar{z}} \psi_+^j + \frac{1}{4} R_{ijkl} \psi_+^i \psi_+^j \psi_-^k \psi_-^l \right). \tag{3.2.1}$$

Here $(z, \bar{z})$ are local coordinates on Σ, $d^2z = -idz \wedge d\bar{z}$, $i, j = 1, \ldots, \dim M$ are target space indices, g_{ij} is the target space metric and R_{ijkl} is its Riemann tensor. Note that here really should be $\Phi^* g_{ij}$, the pullback to the worldsheet of the target space metric, to make this expression well-defined. Here we separated the usual fermion kinetic energy $\overline{\psi} \gamma^\mu D_\mu \psi$ using 2-dimensional γ-matrices* and the component fields $\psi_\pm^i$ of the Dirac spinors ψ^i, which transform as worldsheet fermions, but are target space vectors. The worldsheet Lorentz symmetry, which is just a *global* 2-dimensional rotation, acts as

$$z \longmapsto e^{i\alpha} z, \qquad \psi_\pm \longmapsto \psi_\pm e^{\mp i\alpha/2}. \tag{3.2.2}$$

Note that ψ correctly transforms as a fermion due to the factor $\frac{1}{2}$. To be more precise, we denote the canonical line bundle $K = \Omega^{(1,0)}(M) \cong T^{(0,1)}\Sigma$ of 1-forms on Σ and its conjugate $\overline{K} = \Omega^{(0,1)}(M) \cong \overline{T^{(0,1)}\Sigma} = T^{(1,0)}\Sigma$. Since a 1-form transforms under Lorentz transformations as $dz \longmapsto e^{i\alpha} dz$, looking at the Lorentz transformation rule of ψ, we see that ψ_+ is a section of the square root of $\overline{K}$ and ψ_- is a square root of K. We will denote these square roots by $K^{1/2}, \overline{K^{1/2}} = K^{-1/2}$, which we can think of as being spanned by $\sqrt{dz}, \sqrt{d\bar{z}}$. With this, we see that the correct geometric interpretation is that the fermions ψ are Grassmann sections of the tensor product

$$\psi_+^i \in \Gamma(\Sigma, K^{1/2} \otimes \Phi^*(TM)), \qquad \psi_-^i \in \Gamma(\Sigma, \overline{K^{1/2}} \otimes \Phi^*(TM)), \tag{3.2.3}$$

where $\Phi^*(TM)$ is the pullback of the target space tangent bundle. The covariant derivatives are accordingly defined as the pullback of the Levi-Civita connection on TM, $D_{\bar{z}} \psi_+^i = \partial_{\bar{z}} \psi_+^i + \partial_{\bar{z}} \phi^j \Gamma^i{}_{jk} \psi_+^k$. D_z is defined analogously. The supersymmetry transformations now are

$$\delta\phi^i = i\epsilon_- \psi_+^i + i\epsilon_+ \psi_-^i, \;\; \delta\psi_+^i = -\epsilon_- \partial_z \phi^i - i\epsilon_+ \psi_-^k \Gamma^i{}_{kl} \psi_+^l, \;\; \delta\psi_-^i = -\epsilon_+ \partial_{\bar{z}} \phi^i - i\epsilon_- \psi_+^k \Gamma^i{}_{kl} \psi_-^l,$$

where the parameter ϵ_+ is an anti-holomorphic section of $K^{-1/2}$ and ϵ_- is a holomorphic section of $K^{1/2}$. Note that they have to be (anti)-holomorphic in order to pull them through the covariant derivatives upon variation of the Lagrangian, as mentioned before. Note also that $\partial_z \phi^i$ and $\partial_{\bar{z}} \phi^i$ are 1-forms in the 'active transformation' point of view, hence they are a section of K and $\overline{K}$, which makes the expression consistent.

Now we upgrade M: we suppose it is a complex manifold. This extra structure allows to consistently define patch-wise holomorphic and anti-holomorphic coordinates on M, which are compatible with the transition functions. In particular, this means that we can consistently talk about the components

$$\phi^i \longmapsto \left\{ \phi^i, \phi^{\bar{i}} \right\} \qquad \psi_\pm^i \longmapsto \left\{ \psi_\pm^i, \psi_\pm^{\bar{i}} \right\} \qquad g_{ij} \longmapsto \left\{ g_{i\bar{j}}, g_{\bar{i}j} \right\} \tag{3.2.4}$$

where now $i, j = 1, \ldots, \frac{1}{2} \dim M$ are holomorphic indices and $\bar{i}, \bar{j} = 1, \ldots, \frac{1}{2} \dim M$ are antiholomorphic indices. However, note that the supersymmetry transformations *do not in general preserve this*, since the supersymmetry transformations feature Christoffel symbols. We see that we can consistently define

*Using the generators of $SU(2)$, the Pauli matrices σ^μ, we have $\gamma^0 = \sigma^0, \gamma^1 = -i\sigma^1$. Moreover we use $\psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}$ and $D_z = D_0 + D_1, D_{\bar{z}} = D_0 - D_1$.

$\mathcal{N} = 2$ supersymmetry if M is *Kähler*: in that case the Christoffel symbols are nonzero only for totally holomorphic or anti-holomorphic indices (see also the appendix). In this special case, the action becomes

$$S = \int_\Sigma d^2z \left(\frac{1}{2} g_{i\bar{j}} \partial_z \phi^i \partial_{\bar{z}} \phi^{\bar{j}} + \frac{1}{2} g_{\bar{i}j} \partial_z \phi^{\bar{i}} \partial_{\bar{z}} \phi^j + i g_{\bar{i}j} \psi_-^{\bar{i}} D_z \psi_-^j + i g_{\bar{i}j} \psi_+^{\bar{i}} D_{\bar{z}} \psi_+^j + R_{i\bar{j}k\bar{l}} \psi_+^i \psi_+^{\bar{j}} \psi_-^k \psi_-^{\bar{l}} \right).$$

where we used $g_{i\bar{j}} = g_{\bar{j}i}$ and the symmetries of the curvature tensor (see the appendix). We now double the number of supersymmetries to 4 real supercharges, since 1 Weyl spinor in 2 dimensions has 1 complex degree of freedom, and $\mathcal{N} = 2$. The associated supersymmetry transformations are

$$\begin{aligned} \delta\phi^i &= i\alpha_- \psi_+^i + i\alpha_+ \psi_-^i, & \delta\psi_+^i &= -\tilde{\alpha}_- \partial_z \phi^i - i\alpha_+ \psi_-^k \Gamma^i_{kl} \psi_+^l, \\ \delta\phi^{\bar{i}} &= i\tilde{\alpha}_- \psi_+^{\bar{i}} + i\tilde{\alpha}_+ \psi_-^{\bar{i}}, & \delta\psi_+^{\bar{i}} &= -\alpha_- \partial_z \phi^{\bar{i}} - i\tilde{\alpha}_+ \psi_-^{\bar{k}} \Gamma^{\bar{i}}_{\bar{k}\bar{l}} \psi_+^{\bar{l}}, \\ \delta\psi_-^i &= -\tilde{\alpha}_+ \partial_{\bar{z}} \phi^i - i\alpha_- \psi_+^k \Gamma^i_{kl} \psi_-^l, & \delta\psi_-^{\bar{i}} &= -\alpha_+ \partial_{\bar{z}} \phi^{\bar{i}} - i\tilde{\alpha}_- \psi_+^{\bar{k}} \Gamma^{\bar{i}}_{\bar{k}\bar{l}} \psi_-^{\bar{l}}, \end{aligned}$$

Since the Kähler structure allows for two holomorphic and two anti-holomorphic supersymmetry parameters, this is $\mathcal{N} = (2,2)$ supersymmetry. For completeness, the spinors and parameters are sections

$$\begin{aligned} \psi_+^i &\in \Gamma(\Sigma, \overline{K^{1/2}} \otimes \Phi^*(T^{(1,0)}M)), & \psi_+^{\bar{i}} &\in \Gamma(\Sigma, \overline{K^{1/2}} \otimes \Phi^*(T^{(0,1)}M)), \\ \psi_-^i &\in \Gamma(\Sigma, K^{1/2} \otimes \Phi^*(T^{(1,0)}M)), & \psi_-^{\bar{i}} &\in \Gamma(\Sigma, K^{1/2} \otimes \Phi^*(T^{(0,1)}M)), \\ \alpha_+, \tilde{\alpha}_+ &\in \Gamma(\Sigma, \overline{K^{1/2}}), & \alpha_-, \tilde{\alpha}_- &\in \Gamma(\Sigma, K^{1/2}). \end{aligned}$$

Symmetries

We have given the supersymmetry transformations of the fields generated by the 4 real supercharges, which we will denote as $Q_\pm, \overline{Q}_\pm$, which obey the algebra

$$\{Q_\pm, \overline{Q}_\pm\} = P \pm H, \tag{3.2.5}$$

where P, H are the Euclidean generators of space and time translations on the worldsheet Σ. The supersymmetry variation is written as

$$\delta = i\alpha_- Q_+ + i\alpha_+ Q_- + i\tilde{\alpha}_- \overline{Q}_+ + i\tilde{\alpha}_+ \overline{Q}_-. \tag{3.2.6}$$

Note that the supersymmetry parameters and supercharges sit in the conjugate spinor bundles $K^{1/2}$ and $\overline{K^{1/2}}$, to let δ be invariant under Lorentz transformations: denoting the $SO(2)$-generator of Lorentz transformations by M, this acts on the supercharges as

$$[M, Q_\pm] = \mp Q_\pm, \qquad [M, \overline{Q}_\pm] = \mp \overline{Q}_\pm. \tag{3.2.7}$$

Furthermore, the $\mathcal{N} = (2,2)$ model admits two R-symmetries: the axial and vectorial R-symmetries generated by F_V, F_A. They act only on the spinors:

$$e^{i\alpha F_V} : \left\{ \psi_\pm^i, \psi_\pm^{\bar{i}} \right\} \longmapsto \left\{ e^{-i\alpha} \psi_\pm^i, e^{i\alpha} \psi_\pm^{\bar{i}} \right\}, \qquad e^{i\alpha F_A} : \left\{ \psi_\pm^i, \psi_\pm^{\bar{i}} \right\} \longmapsto \left\{ e^{\mp i\alpha} \psi_\pm^i, e^{\pm i\alpha} \psi_\pm^{\bar{i}} \right\}. \tag{3.2.8}$$

Since the supercharges are spinors too, they transform nontrivially under the R-symmetry:

$$[F_V, Q_\pm] = Q_\pm. \qquad [F_V, \overline{Q}_\pm] = -\overline{Q}_\pm, \qquad [F_A, Q_\pm] = \pm Q_\pm, \qquad [F_A, \overline{Q}_\pm] = \mp \overline{Q}_\pm. \tag{3.2.9}$$

Twisting

As we noted, the supersymmetry parameters were sections of the spinor bundles $K^{1/2}, \overline{K^{1/2}}$, which in general do not admit global non-vanishing sections. In order then to define supersymmetry globally, we need to adjust the theory such that the supersymmetry parameter can be in a generally trivial bundle: we guess that it should be a scalar. To do this, we twist the theory, which amounts to a redefinition of the Lorentz group. From the point of view of the symmetry generators, we define new Lorentz generators as

$$M_A = M + F_V, \qquad M_B = M + F_A. \tag{3.2.10}$$

Then if we define the *topological supercharge* $Q_A = \overline{Q}_+ + Q_-$ and $Q_B = \overline{Q}_+ + \overline{Q}_-$, it is easy to check that $[M_A, Q_A] = [M_B, Q_B] = 0$.

					A-twist		B-twist	
Generators	F_V	F_A	M		$M+F_V$		$M+F_A$	
Group / bundle	$U(1)_V$	$U(1)_A$	$U(1)_E$	L	$U(1)'_E$	L	$U(1)'_E$	L
Q_-, ψ_-	-1	1	1	$K^{1/2}$	0	$\mathbb{C}$	2	K
$\overline{Q}_+, \overline{\psi}_+$	1	1	-1	$\overline{K}^{1/2}$	0	$\mathbb{C}$	0	$\mathbb{C}$
$\overline{Q}_-, \overline{\psi}_-$	1	-1	1	$K^{1/2}$	2	K	0	$\mathbb{C}$
Q_+, ψ_+	-1	-1	-1	$\overline{K}^{1/2}$	-2	$\overline{K}$	-2	$\overline{K}$

Table 1: An overview of $U(1)$-charges and the new bundles after the A-twist and B-twist. The subscript E indicates the Lorentz group.

Performing this twist for all spinors in the theory shows that we get half as much scalar supersymmetries, which enables us to define the supersymmetric theory on an arbitrary curved manifold. Note especially that in twisting, we have to use *global symmetries*. Also note that twisting on flat space does nothing: in that case, we are merely relabeling our symmetry generators, obtaining a scalar and a vector supercharge. But both are globally defined on flat space, hence we look at fields in a different way, but can retain the number of supersymmetries.

Here we glossed over an important detail: we can only twist with the R-symmetries if they remain a symmetry at the quantum level before twisting. Therefore, we need to check whether or not the path integral measure of the $\mathcal{N} = (2,2)$ σ model is invariant under the R-symmetries. To check this, we need to compute the number of zero modes. By complex conjugation of $(D_{\overline{z}}\psi_+)^* = D_z\overline{\psi}_-$, the number of zero modes l_+ for ψ_+ and $\overline{\psi}_-$ is the same. Likewise, l_- is the number of zero modes for ψ_- and $\overline{\psi}_+$. By checking (3.2.8) it is clear that the vector R-symmetry is always preserved. However the path integral measure will not be invariant under the axial R-symmetry: it will transform by $e^{2i(l_+-l_-)\alpha}$. Now we have $l_+ = \dim H^0(K^{1/2} \otimes \Phi^*(T^{(1,0)}M))$, where the H^i denote the sheaf cohomology groups of D_z and $D_{\overline{z}}$. By Serre duality, $H^i(E) = H^{n-i}(K \otimes \overline{E})^*$, we have

$$\begin{aligned}\dim H^1(K^{1/2} \otimes \Phi^*(T^{(1,0)}M)) &= \dim H^0(K \otimes \overline{K}^{1/2} \otimes \Phi^*(T^{(0,1)}M))^* \\ &= \dim H^0(K^{1/2} \otimes \Phi^*(T^{(0,1)}M))^* = l_-.\end{aligned}$$

where * indicates the dual vector space. The Atiyah-Singer index formula tells us that

$$\dim H^0(K^{1/2} \otimes \Phi^*(T^{(0,1)}M)) - \dim H^1(K^{1/2} \otimes \Phi^*(T^{(0,1)}M)) = \int_\Sigma \mathrm{ch}(K^{1/2} \otimes \Phi^*(T^{(1,0)}M))\mathrm{td}(\Sigma)$$

The left-hand side exactly equals the wanted number $l_+ - l_-$. Using

$$\begin{aligned}\mathrm{ch}(K^{1/2} \otimes \Phi^*(T^{(0,1)}M)) &= \mathrm{ch}(K^{1/2})\mathrm{ch}(\Phi^*(T^{(0,1)}M)) = \sqrt{\mathrm{ch}(K)}\mathrm{ch}(\Phi^*(T^{(0,1)}M)) \\ &= \left(1 - \frac{1}{2}c_1(T^{(1,0)\Sigma}\right)\left(d + \Phi^*(c_1(T^{(1,0)}M))\right) \\ &= d + \Phi^*(c_1(T^{(1,0)}M)) - \frac{d}{2}c_1(T^{(1,0)}\Sigma) + \dots, \\ \mathrm{td}(T^{(1,0)}\Sigma) &= 1 + \frac{1}{2}c_1(T^{(1,0)}\Sigma) + \dots,\end{aligned}$$

one straightforwardly finds (keeping only 2-form terms) that

$$l_+ - l_- = \int_\Sigma \Phi^*(c_1(TM)). \tag{3.2.11}$$

Hence we see that while we can always twist by F_V, twisting by F_A is only possible when the target space is Calabi-Yau. Some more details on index theorems and fermion zero modes can be found in appendix ??.

3.3 The A-model

After the A-twist we rewrite the fermions so it is clearer that they have become scalars and vectors:

$$\psi^i_- \longmapsto \chi^i, \qquad \psi^{\bar{i}}_- \longmapsto \psi^{\bar{i}}_z, \qquad \psi^i_+ \longmapsto \psi^i_{\bar{z}}, \qquad \psi^{\bar{i}}_+ \longmapsto \chi^{\bar{i}}. \tag{3.3.1}$$

In this notation, the action for the A-model is

$$S_A = \int_\Sigma d^2z \left(\frac{1}{2} g_{\bar{i}j} \partial_z \phi^i \partial_{\bar{z}} \phi^{\bar{j}} + \frac{1}{2} g_{\bar{i}j} \partial_z \phi^{\bar{i}} \partial_{\bar{z}} \phi^j - i g_{\bar{i}j} \psi^{\bar{i}}_z D_{\bar{z}} \chi^j + i g_{i\bar{j}} \psi^i_{\bar{z}} D_z \chi^{\bar{j}} - R_{i\bar{j}k\bar{l}} \psi^i_{\bar{z}} \psi^{\bar{j}}_z \chi^k \chi^{\bar{l}} \right). \tag{3.3.2}$$

After the A-twist, the supersymmetry parameters $\alpha_-, \tilde{\alpha}_+$ are Grassmann Lorentz scalars while $\alpha_+, \tilde{\alpha}_-$ are Grassmann Lorentz vectors. Setting to zero the latter two, and denoting the scalar ones by $\alpha, \tilde{\alpha}$, the scalar topological supersymmetry variation is $\delta = i\tilde{\alpha}_+ \overline{Q}_+ + i\alpha_- Q_-$. The new supersymmetry transformations become

$$\begin{aligned} &\delta\phi^i = i\alpha\chi^i, \qquad \delta\psi^i_{\bar{z}} = -\tilde{\alpha}\partial_{\bar{z}}\phi^i - i\alpha\chi^k \Gamma^i{}_{kl} \psi^l_{\bar{z}}, \qquad \delta\chi^i = \delta\chi^{\bar{i}} = 0, \\ &\delta\phi^{\bar{i}} = i\tilde{\alpha}\chi^{\bar{i}}, \qquad \delta\psi^{\bar{i}}_z = -\alpha\partial_z\phi^{\bar{i}} - i\tilde{\alpha}\chi^{\bar{k}} \Gamma^{\bar{i}}{}_{\bar{k}\bar{l}} \psi^{\bar{l}}_z. \end{aligned} \tag{3.3.3}$$

We note that the topological supercharge is nilpotent *on-shell*: it is possible to introduce auxiliary fields to get nilpotency off-shell. To get the variation associated to Q_A, we set $\alpha = \tilde{\alpha}$, so that $\delta = i\alpha(\overline{Q}_+ + Q_-) = i\alpha Q_A$. In that case, the nilpotency of Q is trivial. As before, interpreting $\chi^i = d\phi^i$, Q_A acts as the de Rham differential on ϕ and χ. Now we can express the action as

$$S_A = it \int_\Sigma \{Q_A, V\} + \int_\Sigma \phi^*(\omega), \qquad V = g_{\bar{i}j} \left(\psi^{\bar{i}}_z \partial_{\bar{z}} \phi^j + \partial_z \phi^{\bar{i}} \psi^j_{\bar{z}} \right) \tag{3.3.4}$$

and $\omega_{i\bar{j}} = -i g_{i\bar{j}} dx^i \wedge d\overline{x}^{\bar{j}}$ is the Kähler form of M, whose pullback to the worldsheet is $\Phi^*(\omega)$. We also added a coupling constant for localization purposes later (note that we can arbitrarily add Q-exact terms to the Lagrangian at will). The Q_A exact part becomes

$$it \int_\Sigma \{Q_A, V\} = 2t \int_\Sigma d^2z \left(-g_{\bar{i}j} \psi^{\bar{i}}_z D_{\bar{z}} \chi^j + i g_{i\bar{j}} \psi^i_{\bar{z}} D_z \chi^{\bar{j}} - R_{i\bar{j}k\bar{l}} \psi^i_{\bar{z}} \psi^{\bar{j}}_z \chi^k \chi^{\bar{l}} \right). \tag{3.3.5}$$

We see that the A-model is almost topological in the sense described in the previous sector: (3.3.4) is almost Q_A-exact. However, the second term in (3.3.4) only depends on the *homology class* of $\Phi(\Sigma)$ (see chapter C). The consequence is that we can split up the A-model path integral as a sum of the basis elements of $H_2(M, \mathbb{Z})$:

$$Z = \sum_{\beta \in H_2(M,\mathbb{Z})} \exp(-\omega \cdot \beta) \int_{[\phi(\Sigma) \in \beta]} \mathcal{D}\phi \mathcal{D}\chi \mathcal{D}\psi \exp\left(-it \int \{Q_A, V\} \right) \tag{3.3.6}$$

where $\omega \cdot \beta \equiv \int_\beta \omega$. We see that the individual terms in the sum can be regarded as describing a topological field theory. From the A-model action (3.3.2) or from V it is clear that in the limit $\hbar^{-1} = t \longrightarrow \infty$, or by considering fermionic Q-fixed points, the theory *localizes on holomorphic maps* $\partial_z \phi^{\bar{i}} = \partial_{\bar{z}} \phi^i = 0$.

Observables

It is straightforward to find the observables of the A-model. Since inserting the ψs would require worldsheet metric insertions in the path integral, they are not valid local observables, hence all A-model observables are of the form

$$\mathcal{O}_C(x) = C_{i_1 \dots i_k \bar{j}_1 \dots \bar{j}_k}(\phi(x)) \chi^{i_1} \dots \chi^{i_k} \chi^{\bar{j}_1} \dots \chi^{\bar{j}_k}. \tag{3.3.7}$$

Using $\chi^i, \chi^{\bar{i}} \sim d\phi^i, d\phi^{\bar{i}}$, they should be viewed as (k,k)-forms on M. They satisfy $\{Q_A, \mathcal{O}_C\} = \mathcal{O}_{dC}$: we identify the Q_A cohomology $\mathcal{H}(Q_A)$ with the de Rham cohomology $H(M)$ of M. These observations are a specialization of the general structure of observables in twisted theories, a story we will forego here. Our immediate goal is to describe what correlation functions of these observables compute.

A-model correlators and selection rules

After the twist, the number l_χ of χ and $\overline{\chi}$ zero modes is always the same, by complex conjugation of $\not{D}\chi$, likewise for the number of zero modes l_ψ for the ψs. The R_A-anomaly is present if $l_\chi \neq l_\psi$. A slight modification of (??) follows: the χ zero modes are elements in $H^0(\phi^*(TM))$. Now again the Atiyah-Singer index formula gives

$$\int_\Sigma \mathrm{ch}(\phi^*(TM)) \wedge \mathrm{td}(T\Sigma) = \dim H^0(\phi^*(TM)) - \dim H^1(\phi^*(TM)). \tag{3.3.8}$$

By Serre duality $H^1(\phi^*(TM)) = H^0(K \otimes \phi^*(\overline{TM}))^*$, which is the dual to the space of ψ zero modes. Hence, we see that the right-hand side exactly equals what we need:

$$l_\chi - l_\psi = \int_\Sigma \Phi^* c_1(TM) + 2d \int_\Sigma \frac{1}{2} c_1(T\Sigma) = \int_\Sigma \Phi^* c_1(TM) + d(2 - 2g - h) = 2k. \tag{3.3.9}$$

Here we used $\int c_1(T\Sigma) = \chi(\Sigma) = 2 - 2g - h$, the Euler characteristic for open (and closed) worldsheets with h boundary components and $d = \dim M$.

Now consider a A-model correlator $\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle$, which is non-vanishing only when we have enough operator insertions such that we have soaked up all the fermion zero modes. In the generic case, by (3.3.9) we can only consider correlators with $2k$ χ and/or $\overline{\chi}$ insertions, since ψ operators carry a Lorentz index: inserting such fermions would require worldsheet metric insertions, which kill the topological invariance. So we assume we are in the situation that $l_\psi = 0$, such that dim $H^1(\Phi^*(TM)) = 0$. To preserve the vector R-symmetry, we need k χs and k $\overline{\chi}$s. Note that such a correlator has non-trivial axial R-symmetry charge, again we conclude that the axial R-symmetry is spontaneously broken.

Upon localization, the A-model path integral reduces to a sum over holomorphic maps into the target space M, weighted by the worldsheet area $\int_\Sigma \Phi^*\omega$ and classified by the 2-cycle β which Σ is mapped into, as shown in (3.3.6). We define the space of such maps

$$\mathcal{M}_\Sigma(M, \beta) = \{\Phi : \Sigma \to M \mid \Phi \text{ holomorphic}, \Phi_*[\Sigma] = \beta\},$$

which we assume to be a smooth manifold. Then a localized A-model correlation function becomes

$$\langle \mathcal{O}_{C_1}(x_1) \ldots \mathcal{O}_{C_n}(x_n) \rangle = \sum_{\beta \in H_2(M,\mathbb{Z})} e^{-\omega \cdot \beta} \langle \mathcal{O}_{C_1} \ldots \mathcal{O}_{C_n} \rangle_\beta, \tag{3.3.10}$$

where

$$\langle \mathcal{O}_{C_1} \ldots \mathcal{O}_{C_n} \rangle_\beta = \int_{\mathcal{M}_\Sigma(M,\beta)} \mathrm{ev}_1^* \omega_1 \wedge \ldots \wedge \mathrm{ev}_n^* \omega_n, \tag{3.3.11}$$

using $\mathrm{ev}_i : \mathcal{M}_\Sigma(M, \beta) \to M, \Phi \mapsto \Phi(x_i)$. In a special case, this leads to something nice: suppose $\{D_i\}$ is a collection of submanifolds that intersect transversely in M and have $\sum_i \dim_\mathbb{R} D_i = \dim_\mathbb{R} M$, then choosing operators $\mathcal{O}_{C_i}$ such that C_i is the Poincaré dual* to D_i gives the correlator

$$\langle \mathcal{O}_{C_1}(x_1) \ldots \mathcal{O}_{C_n}(x_n) \rangle = \sum_{\beta \in H_2(M,\mathbb{Z})} e^{-\omega \cdot \beta} n_{\beta, D_1, \ldots, D_n}, \tag{3.3.13}$$

*The Poincare dual η_S to a submanifold S is determined by the condition that

$$\int_S \iota^* \omega = \int_M \omega \wedge \eta_S. \tag{3.3.12}$$

Note that the degree of $\eta_S =$ codim S and η_S has delta-function support on S. In Bott, this is done for a closed oriented submanifold, we will just use the generalization to a non-compact submanifold.

where $n_{\beta,D_1,\ldots,D_n}$ is the number of holomorphic maps that obey $\Phi(\Sigma) = \beta$ and $\Phi(x_i) \in D_i, \forall i$. These numbers are called the *Gromov-Witten invariants*. When $\beta = 0$, so $\Phi(\Sigma)$ is a point, we have $\mathcal{M}(M,0) \cong M$ and $\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle = \int_M \omega_1 \wedge \ldots \wedge \omega_n = \#(D_1 \cap \ldots \cap D_n)$ is the classical intersection number. This immediately gives the interpretation of the higher order correlators: they give a quantum deformation of the classical intersection number, given by worldsheet instantons in M.

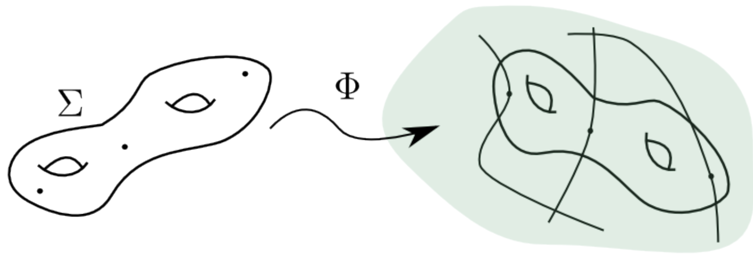


Figure 2: Worldsheet embeddings with operator insertions at marked points mapped to transversal submanifolds in the target space.

3.4 Topological branes

So far we have discussed the topological theories with closed worldsheet. Naturally, these models generalize to the case with open worldsheet, which will important to us in the coming chapters. Our main aim is to discuss how the open worldsheet can couple to topological branes. The canonical reference for this is [6], in which the co-isotropic brane was described for the first time.

We recall that we defined the topological supercharges $Q_A = \overline{Q}_+ + Q_-$ and $Q_B = \overline{Q}_+ + \overline{Q}_-$. For convenience, we take a flat worldsheet $\Sigma = \mathbb{R} \times (-\infty, 0]$ and consider the supersymmetric σ model with $\Phi : \Sigma \to M$, equipped with superpotential W and whose target space M is Kähler. M carries a Kähler metric g and Kähler form ω, moreover the worldsheet embedding Φ maps the boundary $\partial\Sigma$ into a submanifold $N \subset M$. This model is called the *Landau-Ginzburg model* and we will encounter this again in chapter 6. Our goal is to find under what conditions N can be viewed as a topological D-brane. We'll use worldsheet coordinates $(x^0, x^1) \in \mathbb{R} \times (-\infty, 0]$ with $w^\pm = x^0 \pm x^1$ such that $\partial_\pm = \partial_0 \pm \partial_1$. The action reads

$$S_{LG} = \int_\Sigma d^2w \left(2g_{i\bar{j}}\bar{\partial}\phi^i \partial\phi^{\bar{j}} + 2g^{i\bar{j}}\partial_i W \partial_{\bar{j}}\overline{W} + \psi^i_+ \psi^j_- D_i \partial_j W + \psi^{\bar{i}}_+ \psi^{\bar{j}}_- D_{\bar{i}} \partial_{\bar{j}} \overline{W} \right)$$
$$+ \int_\Sigma d^2w \left(\frac{i}{2} g_{\mu\nu} \psi^\mu_+ D_{\overline{w}} \psi^\nu_+ + \frac{i}{2} g_{\mu\nu} \psi^\mu_- D_w \psi^\nu_- + R_{i\bar{j}k\bar{l}} \psi^i_+ \psi^{\bar{j}}_+ \psi^k_- \psi^{\bar{l}}_- \right)$$

and the four supercharges $Q_\pm$ and $\overline{Q}_\pm$ are the worldsheet x^1 integrals of the supercurrents $G^0_\pm$

$$G^0_\pm = g_{i\bar{j}} \partial_\pm \phi^{\bar{j}} \psi^i_\pm \mp \frac{i}{2} \psi^{\bar{i}}_\mp \partial_{\bar{i}} \overline{W}, \qquad G^1_\pm = \mp g_{i\bar{j}} \partial_\pm \phi^{\bar{j}} \psi^i_\pm - \frac{i}{2} \psi^{\bar{i}}_\mp \partial_{\bar{i}} \overline{W}, \tag{3.4.1}$$

$$\overline{G}^0_\pm = g_{i\bar{j}} \partial_\pm \phi^i \psi^{\bar{j}}_\pm \pm \frac{i}{2} \psi^i_\mp \partial_i W, \qquad \overline{G}^1_\pm = \mp g_{i\bar{j}} \partial_\pm \phi^i \psi^{\bar{j}}_\pm + \frac{i}{2} \psi^i_\mp \partial_i W. \tag{3.4.2}$$

These follow directly from Noether's theorem applied to supersymmetry. Then to preserve $\mathcal{N} = 1$ supersymmetry we need the variation of the action

$$\delta S = \int_\Sigma d^2x \delta X (\text{bulk EoM}) + \int_{\partial\Sigma} dx^0 \delta X (\text{boundary terms}) \tag{3.4.3}$$

to vanish, where X represents the bosonic and fermionic fields of the theory. Combining the holomorphic and anti-holomorphic indices into a single index I, the boundary variation δX(boundary terms) is

$$g_{IJ} \delta\phi^I \partial_1 \phi^J = 0, \qquad g_{IJ} \left(\psi^I_- \delta\psi^J_- - \psi^I_+ \delta\psi^J_+ \right) = 0, \tag{3.4.4}$$

at $\partial\Sigma$. Since the $\mathcal{N}=1$ supersymmetry transformations are given by

$$\delta\phi^I = i\epsilon\left(\psi_+^I + \psi_-^I\right), \qquad \delta\psi_\pm^I = -\epsilon\partial_\pm\phi^I \mp \epsilon g^{IJ}\partial_J \text{Im}\, W \pm i\epsilon\Gamma^I_{JK}\psi_+^J\psi_-^K, \tag{3.4.5}$$

the boundary variation contribution is

$$\delta S|_{\partial\Sigma} = \frac{i\epsilon}{2}\int_{\partial\Sigma} dx^0\left(-g_{IJ}\partial_0\phi^J\left(\psi_-^I - \psi_+^I\right) - g_{IJ}\partial_1\phi^J\left(\psi_-^I + \psi_+^I\right) - \frac{i}{2}\left(\psi_-^I + \psi_+^I\right)\partial_I(W - \overline{W})\right). \tag{3.4.6}$$

Realizing that ϕ maps $\partial\Sigma$ into N, we get that $\delta\phi^I$ must be tangent to N, now it is easy to see from (3.4.4) and (3.4.6) that

$$t_b = \partial_0\phi^I \text{ is tangent to } N, \qquad t_f = i\left(\psi_-^I + \psi_+^I\right) \text{ is tangent to } N, \tag{3.4.7}$$

$$n_b = \partial_1\phi^I \text{ is normal to } N, \qquad n_f = i\left(\psi_-^I - \psi_+^I\right) \text{ is normal to } N \tag{3.4.8}$$

and that Im W has to be constant on a connected component of N.

In addition, we want to preserve $\mathcal{N}=2$ supersymmetry. For A-type supersymmetry, the condition is that the space component of the supercurrent vanishes at the boundary: $\overline{G}^1_+ + G^1_- = 0$, for B-type supersymmetry it is the obvious analogous condition. Since we shall only consider the open A-model in the hereafter, we only look at the first case. Let us first consider the case that there is no B-field on the worldsheet or a non-zero gauge field on the brane N. From (3.4.2) we find that

$$\begin{aligned}\overline{G}^1_+ + G^1_- &= \frac{1}{2}\left(g(n_b,t_f) - g(t_b,n_f)\right) - \frac{i}{2}\left(\omega(t_b,t_f) - \omega(n_b,n_f)\right) + \frac{i}{2}t_f^I\partial_I \text{Im}\, W + \frac{1}{2}n_f^I\partial_I \text{Re}\, W \\ &= \frac{i}{2}\left(\omega(n_b,n_f - \omega(t_b,t_f)\right) + \frac{1}{2}n_f^I\partial_I \text{Re}\, W.\end{aligned} \tag{3.4.9}$$

Since the vectors $t_{b,f}, n_{b,f}$ are arbitrary, the last three terms have to vanish individually. Vanishing of $\omega(t_f,t_b)$ means that N is isotropic with respect to the symplectic form ω, while $\omega(n_b,n_f)=0$ implies that N is co-isotropic. This means that N is a Lagrangian submanifold in M. The third term vanishes automatically. This follows from considering the A-model supersymmetry transformations: these read for a holomorphic index

$$\delta\phi^i = \epsilon\left(\psi_-^I - \psi_+^i\right) + i\epsilon\left(\psi_-^i + \psi_+^i\right) = \epsilon t_f - i\epsilon n_f. \tag{3.4.10}$$

Now $\delta\phi^i$ must be tangent to N, but tells us that in_f should be tangent to N. Comparing to the constraints in (??) and (??), we learn that multiplication by i turns any holomorphic normal vector into a tangent vector. Since we know that $g^{IJ}\partial_J$Im W is a holomorphic normal vector to N, $ig^{IJ}\partial_J$Im W is tangent to N. Hence also $g^{IJ}\partial_J W$ is tangent, so $g^{IJ}\partial_J$Re W must be tangent to N. Therefore, the third term in (3.4.9) vanishes.

Turning on a B-field on the worldsheet has the same effect as turning on a gauge field on N. In the latter case, the gauge field shows up as a boundary contribution to the action

$$\int_{\partial\Sigma} A_I d\phi^I. \tag{3.4.11}$$

Its variation is given by $\int_{\partial\Sigma} dx^0\delta\phi^I\partial_0\phi^J F_{IJ}$, where $F_{IJ} = \partial_{[I}A_{J]}$ is the curvature of A. With this extra boundary contribution, it is straightforward to check that the boundary conditions for $\mathcal{N}=1$ supersymmetry on the fields are modified to

$$\begin{aligned} t_b = \partial_0\phi^I &\text{ is tangent to } N, \qquad t_f = \psi_-^I + \psi_+^I \text{ is tangent to } N, \\ n_b = \partial_1\phi^I + g^{IM}F_{MN}\partial_0\phi^N &\text{ is normal to } N, \qquad n_f = \psi_-^I - \psi_+^I - g^{IM}F_{MN}t_f^N \text{ is normal to } N.\end{aligned}$$

Here we left the inclusion $\iota : TN \to TM$ (for the indices on F) implicit, just as the metric g, which pedantically is g restricted to N. Again, Im W must be constant on N. The condition for $\mathcal{N} = 2$ supersymmetry is modified to

$$\overline{G}^1_+ + G^1_- = \frac{i}{2}\left((\omega + F\omega^{-1}F)(t_b, t_f) - \omega(n_b, n_f)\right) + \frac{i}{2}\omega^{-1}(gn_b, FT_f) + \frac{i}{2}\omega^{-1}(gn_f, Ft_b) + \frac{i}{2}\left(n_f + \frac{i}{2}g^{-1}Ft_f\right)^I \partial_I \mathrm{Re}\, W. \tag{3.4.12}$$

Here we have dropped most indices for visual clarity. The term including $\partial_I \mathrm{Re}\, W$ vanishes for the same reasons stated earlier. We now denote by

$$(TN)^\perp = \{v \in TM \mid \omega(v, w) = 0, \forall w \in TN\} \tag{3.4.13}$$

the orthogonal complement with respect to ω. As before, the six terms in (3.4.12) must vanish individually. Vanishing of $\omega(n_b, n_f)$ implies that N is co-isotropic, so $(TN)^\perp \subset TN$. Furthermore,

$$\omega^{-1}(gn_f, Ft_b) = 0 \Rightarrow F = 0 \text{ on } (TN)^\perp \times TN$$
$$(\omega + F\omega^{-1}F)(t_b, t_f) = 0 \Rightarrow \omega + F\omega^{-1}F = 0 \text{ on } TN/(TN)^\perp.$$

The last condition means that

$$(\omega^{-1}F)^2 = -1, \tag{3.4.14}$$

so that $\omega^{-1}F = J$ is an almost complex structure. It can be shown that J actually is integrable: it is an honest complex structure. To show this, the Nijenhuis tensor for J should vanish. This was proven in [6]. The idea of the proof for Lagrangian branes uses that both ω and F are symplectic. Since $\omega^{-1}F$ has eigenvalues $\pm i$ by (3.4.14) $\omega + rF$ is symplectic for any real r and hence invertible. It follows that ω^{-1} and F^{-1} are compatible Poisson structures* and by the fundamental theorem of bihamiltonian geometry it follows that the Nijenhuis tensor of $\omega^{-1}F$ vanishes.

To summarize, we see that in general open A-type worldsheets can end on a coisotropic submanifold of M, on which there is a gauge field with non-vanishing curvature. It turns out by more careful consideration (see [6]) of the form of ω^{-1} that $\dim_{\mathbb{R}} M - \frac{1}{2}\dim_{\mathbb{R}} N$ must be even. Note that this implies that for M a Kähler manifold the co-isotropic brane of maximal dimension, namely $\dim_{\mathbb{R}} M$, always is an admissible A-brane. This A-brane is called the *canonical co-isotropic brane*, which we shall denote in the hereafter by $\mathcal{B}_{cc}$. Likewise, we shall denote a Lagrangian A-brane by $\mathcal{B}_{\mathcal{L}}$.

The category of A-branes

It is known that A-branes sit in a category of their own: the Fukaya category $\mathcal{F}^0(M)$. However, the understanding of this category is still far from complete. The main feature of $\mathcal{F}^0(M)$ is that its morphisms come equipped with an A^∞-structure. The morphisms can be represented by open A-strings* with disk worldsheets, which end on the A-branes. We shall be rather descriptive here, as we will not need the technical details, which can be found in [7].

$\mathcal{F}^0(M)$ is derived from another category called $\mathcal{F}(M)$. Recall that an A-brane is characterized by its support, a coisotropic submanifold $\mathcal{B}$, and a vector bundle $E \to \mathcal{B}$. Therefore we represent objects in $\mathcal{F}(M)$ as a pair $O = (\mathcal{B}, E)$. For any pair of objects O_i, O_j, we then have an abelian group of morphisms $\mathrm{Hom}(O_i, O_j)$ that carry an A^∞-structure. An A^∞-structure is a souped-up version of a differential graded algebra: it is a $\mathbb{Z}$-graded algebra, with a degree 1 map m_1, which squares to 0, analogous to the de Rham differential. However, this algebra also contains higher degree maps m_k, which satisfy a system of non-linear conditions. Let us restrict ourselves to Lagrangian A-branes O_1, O_2. When two such branes

*A Poisson structure is a skew-symmetric map $\{.,.\}$ that satisfies the Jacobi identity and is a derivation in its first argument. The simplest example is the Poisson structure induced by a symplectic form ω as $\{f, g\} = \omega_{ij}\partial_i f \partial_j g$, as on any classical phase space.

*We abuse language here: we mean just the open A-model, not the full topological string.

intersect transversally in a point $p = \mathcal{B}_1 \cap \mathcal{B}_2$, one can define $\mathrm{Hom}(O_1, O_2) = \mathrm{Hom}((E_1)_p, (E_2)_p))$, as the space consisting of holomorphic disks that map (parts of) the boundary S^1 into $\mathcal{B}_1$ and $\mathcal{B}_2$. More generally, when $n+1$ Lagrangians O_i intersect transversally, one can express the higher degree maps m_k in terms of holomorphic disks whose boundary lies on the given Lagrangians. The morphisms described just now are in general not associative, however their cohomologies are. So by taking the bottom cohomology H^0 of all Homs (which each have the structure of a complex) in $\mathcal{F}(X)$, one gets a good category $\mathcal{F}^0(X)$.

The reasons that $\mathcal{F}^0(M)$ is not completely understood are, for instance, the problems that arise when the branes intersect non-transversely. One also has to deal with coisotropic A-branes, moreover, the exact contents of $\mathcal{F}^0(M)$ is not entirely clear. Finally, the moduli space of holomorphic disk embeddings in X has a codimension-1 boundary, given by the bubbling off of holomorphic disks.* The consequence of this bubbling is that m_1 is not nilpotent anymore, and the A^∞ structure becomes obstructed.

This story is tightly intertwined with mirror symmetry. There are B-branes in the open B-model, which turn out to be complex submanifolds with vector bundle. In particular, B-branes are not equivalent to A-branes and sit in their own category. The space of morphisms is also different, since the B-model localizes on constant maps into M. One can show that the correct category for the B-branes is $\mathcal{D}^b(M)$, the derived category of coherent sheaves on M, which is better understood than the Fukaya category. To any Calabi-Yau manifold M, one can naturally associate $\mathcal{D}^b(M)$ and $\mathcal{F}^0(M)$: the homological mirror symmetry conjecture now posits that these categories are equivalent. In full generality, a rigorous proof is not yet known.

*Bubbling very roughly means the following. Consider the moduli space of holomorphic curves $\Sigma \to M$ that represent the class $\beta \in H_2(M, \mathbb{Z})$, with genus g and n marked points. In symplectic geometry, it can happen that a holomorphic map $f : \mathbb{C} \to M$ with finite area can, under the right circumstances, be extended to a holomorphic map $\tilde{f} : S^2 = \mathbb{C} \cup \{\infty\} \to M$ by removal of singularities. This means that given a sequence of holomorphic curves J_n in the moduli space $\mathcal{M}_{g,n}(\Sigma, \beta)$, it can happen that the limit J_∞ of this sequence is still an honest holomorphic curve in $\mathcal{M}_{g,n}(\Sigma, \beta)$, while having local singularities. Namely, there can be parts of the curve J_∞ that have degenerated into a sphere that has transversal intersection with the rest of the curve: the sphere has *bubbled off*. In this case, the sphere and the rest of J_∞ are connected only by a single node.

4

Exotic integration cycles

We now come to a key application of Morse theory to field theory: we use the Morse flow to find an alternative integration cycle for path integrals. This technique exploits the simple observation that along downward flow, the Morse function h strictly decreases. This property points out suitable integration cycle on which one can obtain formal convergence of highly oscillatory integrals. We will also discuss how we can combine this technique with localization in supersymmetric σ-models. These ideas will be the key to the new dualities discussed in chapter 6 and 8. A reference for this material is [9].

4.1 Morse functions and gradient flow

We first set some terminology. Consider a compact manifold M and a scalar function $h : M \longrightarrow \mathbb{R}$. A *Morse function* is a function h that has isolated critical points $p \in M$ where

$$\left(\frac{\partial h}{\partial x^i}\right)_p = 0, \quad \forall i. \tag{4.1.1}$$

Note that a Morse function only has a finite number of critical points on a compact manifold M.* If M is compact, h also attains its maximum and minimum. If M is not compact, we will *assume* the Morse function still has only a finite number of critical points. For such a Morse function, we can define its *Morse index* as follows: consider the matrix of second derivatives, the Hessian

$$H_{ij} = \left(\frac{\partial^2 h}{\partial x^i \partial x^j}\right)_p \tag{4.1.2}$$

of h in local coordinates at a critical point p, then

$$\text{the Morse index } \mu(p) \text{ is the number of negative eigenvalues of } H_{ij}. \tag{4.1.3}$$

It is a result that Morse functions lie dense in all smooth functions: hence a large class of functions are Morse. We shall first only consider Morse functions with *non-degenerate critical points* where the Hessian has no zero eigenvalues, and indicate later what adjustments should be made in the degenerate case. We denote the number of critical points of index k as N_k.

Now the gradient of the Morse function h will define flow lines that start and end at the critical points of h. This is immediate: the gradient of h defines a vector field on M, which in turn defines integral curves or flow lines by the flow equation. Using local coordinates x^i and a choice of metric g_{ij} on M and a flow parameter s, flow lines are described by a map $y : \mathbb{R} \longmapsto M$ that is a solution to the differential equation

$$\frac{\partial y^i}{\partial s} = \mp g^{ij} \frac{\partial h}{\partial x^j} \tag{4.1.4}$$

which are called the *downward (-) and upward (+) flow equation*. The analysis of the space of solutions of this equation and their behavior forms the starting point of Morse theory. A choice of Morse function h and metric g is called a *Morse-Smale pair* (h, g).

*The set of critical is discrete since the critical points are isolated. Since we can find an open neighborhood around any critical point p, by compactness of M it follows that there are only finitely many critical points.

The Morse lemma states that locally around p we can choose normal coordinates centered at p such that h can be written as

$$h = h_0 - \sum_{i=1}^{\mu(p)} c_i w_i^2 + \sum_{i=\mu(p)+1}^{\dim M} c_i w_i^2 + \mathcal{O}(w^3), \qquad g_{ij} = \delta_{ij} + \mathcal{O}(w^2) \tag{4.1.5}$$

at p. In these coordinate, the downward flow equation becomes simply

$$\frac{dw^i}{ds} = -c_i w^i, \qquad \text{(no summation)}, \qquad w^i(s) = r^i \exp\left(-c_i s\right). \tag{4.1.6}$$

This solution is in general only valid for some finite flow time, after which the solution has to be extended in a new coordinate patch. We shall assume that the flow can always be extended for sufficiently long flow times. Note that if $\frac{\partial w^i}{\partial s} = 0$ at some finite flow time s, the flow equation implies that the flow will remain at that critical point for all s. We conclude that flow lines can only interpolate between critical points at $s = \pm\infty$. Note that in general, there can be Morse flow between critical points, provided the Morse function has suitable behavior. We shall come back to this later.

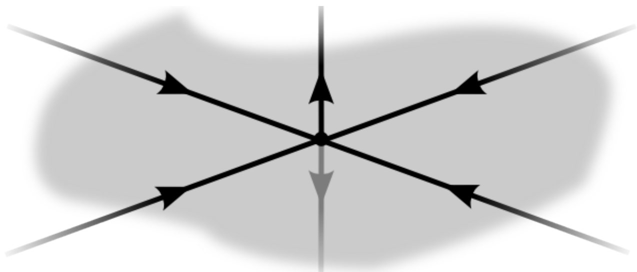


Figure 3: Downward and upward flow from a critical point.

If the flow starts at p at $-\infty$, then necessarily $r^i = 0$ whenever $e_i > 0$. But the number of negative e_i equals the Morse index $\mu(p)$, so we are left with $\mu(p)$ unconstrained r^i. Hence, the family of solutions that start at p is $\mu(p)$-dimensional.

4.2 Exotic integration cycles: a 0-dimensional example

Suppose Φ are some fields and we have an action $S(\Phi)$, then the partition function of the theory is determined formally by the path integral, in Lorentzian signature:

$$Z = \int_{\mathcal{C}} \mathcal{D}\Phi \exp iS(\Phi). \tag{4.2.1}$$

In general, $S(\Phi)$ is a polynomial in Φ and has a positive definite real part. Our goal will be to find an alternative integration cycle $\mathcal{C}'$ so that Z can be expressed as an integral over $\mathcal{C}'$. Normally $\mathcal{C}$ is a trivial cycle: we have to integrate over *all* field configurations Φ; it is clear that to find a different cycle, we first need to 'create more room'. We shall do this by complexifying all the objects in the path integral, which doubles the number of dimensions we can work in, upon which we find suitable middle-dimensional cycles in the complexified space to find $\mathcal{C}'$.

Consider the 0-dimensional oscillatory path integral whose action is the *Airy function* $S(\lambda, x)$, defined for real λ:

$$Z = \int_{\mathcal{C}_{\mathbb{R}}=\mathbb{R}} dx \exp S(\lambda, x), \quad S(\lambda, x) = i\lambda\left(\frac{x^3}{3} - x\right).$$

Suppose we want to analytically continue it to complex λ: this cannot be done arbitrarily, since if Im $\lambda \neq 0$, the action $S(\lambda, x)$ will always diverge to $+\infty$ at either $x \longrightarrow +\infty$ or $x \longrightarrow -\infty$, depending on the sign of Im λ. Hence, to extend the integral to complex λ, we consider the following.

We complexify $x \longrightarrow z$, upon which there are 3 'good' regions at infinity in the complex z-plane where $\mathcal{S}(\lambda, z) \longrightarrow -\infty$, so where the integrand of the path integral falls off exponentially. To be explicit: writing $\lambda = |\lambda| \exp i\theta_\lambda$ and $z = |z| \exp i\theta_z$ the dominant contribution towards infinity comes from

$$\text{Re } i\lambda z^3 = \text{Re}\left[iC\left(\cos(\theta_\lambda + 3\theta_z) + i\sin(\theta_\lambda + 3\theta_z)\right)\right] = -C\sin(\theta_\lambda + 3\theta_z), \qquad C \geq 0. \tag{4.2.2}$$

θ_λ is fixed, and we see clearly that there are three intervals for θ_z where Re $i\lambda z^3 \longrightarrow -\infty$. Since $\mathcal{S}(\lambda, z)$ is a polynomial with positive powers of z, the integrand $\exp \mathcal{S}$ has no poles and we are free to deform the integration cycle $\mathcal{C}_\mathbb{R}$ to $\mathcal{C}'$ to make the path integral convergent. This happens when $\mathcal{C}'$ connects two 'good' regions. If we keep the end points $\mathcal{C}'$ in the same region as the corresponding end points of $\mathcal{C}_\mathbb{R}$, the value of the path integral does not change. This follows easily by Cauchy's theorem: suppose two paths C_1, C_2 that extend to infinity are related by a continuous deformation and are oriented parallel-wise, and let $f(z)$ be a function that has no poles in the region enclosed by C_1 and C_2. Moreover, suppose that $f(z)$ dies off sufficiently fast such that in closing up C_1, C_2 at infinity, we do not get any contributions there (this should be done in a suitable limiting procedure). Then

$$\oint_{C_1} f(z)dz - \oint_{C_2} f(z)dz = \oint_{C_1 - C_2} f(z)dz = 0 \tag{4.2.3}$$

so we see that the two integrals over C_1 and C_2 coincide. Here we should think of the sum $\mathcal{C}_1 - \mathcal{C}_2$ as a union of cycles taking the orientation into account.

Extending this argument to the situation with the Airy function, we can choose three cycles $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ that connect two subsequent 'good' regions. By holomorphicity of $\exp \mathcal{S}(\lambda, z)$ it follows immediately that

$$\int_{\mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3} \exp \mathcal{S}(\lambda, z)dz = 0. \tag{4.2.4}$$

This is also clear from the fact that $\mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3$ can be deformed into a closed contour. If $\mathcal{C}_1$ is a continuous deformation of $\mathcal{C}_\mathbb{R}$ whose endpoints lie the same 'good' region and has the same orientation, we see that

$$\int_{\mathcal{C}_\mathbb{R}} \exp \mathcal{S}(\lambda, z)dz = -\int_{\mathcal{C}_1} \exp \mathcal{S}(\lambda, z)dz = \int_{\mathcal{C}_2 + \mathcal{C}_3} \exp \mathcal{S}(\lambda, z)dz. \tag{4.2.5}$$

The minus sign comes from the orientation of $\mathcal{C}_1$ relative to $\mathcal{C}_\mathbb{R}$. We will call the alternative cycle $\mathcal{C}_2 + \mathcal{C}_3$ an *exotic integration cycle*. We can rephrase the last observation a bit by interpreting the cycles $\mathcal{C}_i$ as generators of the *relative homology* $H_1(\mathbb{C}, \mathbb{C}_{<T})$ where $X_{<T} = \{z \in \mathbb{C} : \text{Re } \mathcal{S}(\lambda, z) < -T\}$, which is the homology of cycles with endpoints in the 'good' regions, one of the components of $\bigcap_{\{T : T > -\infty\}} X_{<T}$. We should think of $H_1(\mathbb{C}, \mathbb{C}_{<T})$ as the equivalence set of 'good' integration cycles modulo smooth deformations that keep the endpoints of a 'good' cycle in the same 'good' region.

Let us now see how we can reproduce such an exotic integration cycle in a more intrinsic way. To do this, we use Morse theory in the complex setting, where is also called Picard-Lefschetz theory. We regard the real part of $\mathcal{S}(\lambda, z)$ as a Morse function $h = \text{Re } \mathcal{S}(\lambda, z)$. This is a smooth function (it is the real part of a function holomorphic in z) which has two isolated critical points at $z = \pm 1$, which have both Morse index 1. At those points, we have

$$\mathcal{S}_\pm = \mp \frac{2i\lambda}{3}, \qquad h_\pm = \pm \frac{2\text{Im } \lambda}{3}. \tag{4.2.6}$$

Note that the critical points of h are the same as those of the uncomplexified action $S(\lambda, x)$, since by the Cauchy Riemann equations, we have for the holomorphic function $\mathcal{S}$: $\frac{\partial \text{Re } \mathcal{S}}{\partial \text{Re } z} = \frac{\partial \text{Im } \mathcal{S}}{\partial \text{Im } z}$.

We now make an observation on Morse flow. Consider a general n complex-dimensional complex manifold M, and let h be the real part of a complexified polynomial, defined in term of local complex coordinates $z^i, \bar{z}^i$ on M. A flow line generated by h is determined by the Morse flow equation, which in local real coordinates w^i and a metric g_{ij} on $\mathbb{C}^n$ reads

$$\frac{dw^i}{ds} = -g^{ij}\frac{\partial h}{\partial w^j} \tag{4.2.7}$$

where s is a flow parameter. Multiplying both sides by the flow speed, we obtain

$$\frac{dw^i}{ds}\frac{\partial h}{\partial w^i} = \frac{dh}{ds} = -g^{ij}\frac{\partial h}{\partial w^j}\frac{\partial h}{\partial w^i} \leq 0. \tag{4.2.8}$$

If $\frac{\partial h}{\partial w^i} \neq 0$, the rightmost term is always negative, by positivity of the metric. This means that if the flow line does not interpolate between critical points, as $s \longrightarrow \infty$, h will always decrease to $-\infty$. This is exactly the behavior we wanted on our exotic integration cycle. Also, along downward flow lines, the maximum of h is attained at p, since h is strictly decreasing along downward flows.

Downward flow and complexification

Since here our Morse function h is the real part of the complexification $\mathcal{S}$ of a real polynomial, by the Morse lemma, locally at p, h is always of the form

$$\mathcal{S} = \mathcal{S}_0 + \sum_{i=1}^{n} c_i z_i^2 + \mathcal{O}(z^3) \Rightarrow h = \text{Re}\,\mathcal{S} = \text{Re}\,\mathcal{S}_0 + \sum_{i=1}^{n} c_i(x_i^2 - y_i^2) + \mathcal{O}(z^3), \tag{4.2.9}$$

so we find immediately that such h always have critical points with Morse index n, which is exactly the dimension of the original real space we started out with. Combined with our earlier observation, we see that for non-degenerate critical points p, the unstable manifold $\mathcal{C}_p$ always is an real n-dimensional cycle or in other words: it is a middle-dimensional cycle.

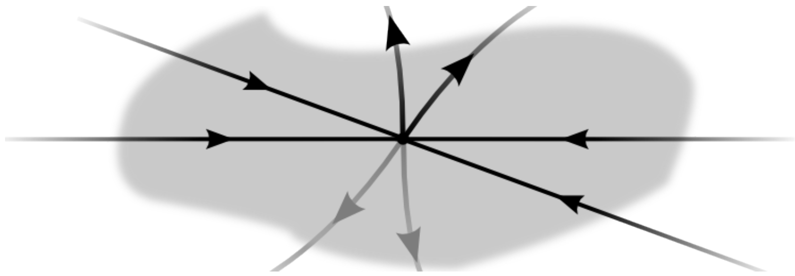


Figure 4: Equal numbers of downward and upward flow directions when the Morse function h comes from a complexification.

Perfect Morse functions

$\mathbb{C}^n$ is not compact, so our choices for h will generically be unbounded from above and below. In this situation, h generates the relative homology $H_n(\mathbb{C}^n, \mathbb{C}^n_{<T})$. A generalization of the weak Morse inequality (C.1.1) now tells us that the rank of $H_n(M, M_{<T})$ is at most the sum of Morse indices of h. If the Morse function does not have pairs of critical points that differ ± 1 in Morse index, there actually is equality. In this case, h is a *perfect Morse function* and the weak Morse inequality is saturated: $b_k(M) = N_k(M)$.*
This immediately tells us that Morse functions that are a complexification are perfect. For instance, for the Airy function, the Morse indices are both 1, and we indeed find that the rank of the relative homology for the Airy function is 2.

*This reflects the fact that only instantons between critical points that had adjacent Morse index can lift the energy of the ground states associated to the critical points, hence removing states from the cohomology of ground states. For this, see appendix C.2.3.

Decomposition in Lefschetz thimbles

Now all the critical points p_i of h generate downward Morse flows, which define middle-dimensional cycles $\mathcal{C}_{p_i}$. In the complex setting as above, these cycles are called *Lefschetz thimbles*. We can then decompose the exotic integration cycle $\mathcal{C}$ as

$$\mathcal{C} = \sum_i n_i \mathcal{C}_{p_i}, \qquad n_i \in \mathbb{Z}. \tag{4.2.10}$$

The coefficients n_i have to be determined intrinsically. Geometrically, as a first guess one might want to use intersection products between $\mathcal{C}$ and the $\mathcal{C}_{p_i}$. However, this is not possible: going back to the Airy function, the intersection between two 1-cycles should be a 0-cycle in $H_0(\mathbb{C}, \mathbb{C}_{<T})$. But any such 0-cycle (a set of points) is always deformable into $\mathbb{C}_{<T}$, so we can always arrange for it to have zero intersection with $\mathcal{C}$. But since the cycles $\mathcal{C}_{p_i}$ are elements in $H_n(M, M_{<T})$, which is a vector space, we can use its dual space to determine the n_i. It follows from Morse theory that the dual space consists of the cycles defined by upward flow. Upward flows are solutions to

$$\frac{dw^i}{ds} = e_i w^i \qquad \text{(no summation)}. \tag{4.2.11}$$

which is the upward Morse flow equation in a local neighborhood of a critical point p. Following the same logic as above, we require that the solution approaches p at flow time $s \longrightarrow -\infty$, it follows that a critical point with Morse index n generates a cycle $\mathcal{K}_p$ of dimension n (in general, if p has Morse index k, $\mathcal{K}_p$ has dimension $2n-k$). The cycles $\mathcal{K}_p$ sit in a different homology group, which we call $H^n(M, M^T)$, where M^T is the subset of M where $h \geq T$. Since the two homology groups are of complementary dimension, there is a natural intersection pairing between them. For the moment, we only consider the case where there are no flows between two distinct critical points. In that case, it follows that the only point where an upward $\mathcal{C}_p$ and downward flow $\mathcal{K}_p$ intersect are at the critical point p. Moreover, by the properties of Morse-Smale theory, they always intersect transversally. Therefore, the natural pairing is such that

$$\langle \mathcal{K}_p, \mathcal{C}_q \rangle = \delta_{pq} \implies n_i = \langle \mathcal{C}, \mathcal{K}_{p_i} \rangle \tag{4.2.12}$$

Note that with this procedure, it is possible to decompose any cycle $\mathcal{C}$ in terms of the Lefschetz thimbles. Now let us apply this to the important case that h is of the form

$$h = \text{Re}\, itg(x_1, \ldots, x_n), \qquad g \text{ is polynomial.} \tag{4.2.13}$$

In our applications, h will always be of this form. Suppose that t is real: what is the generic decomposition of $\mathcal{C}_\mathbb{R}$ into Lefschetz thimbles? Observe that on the set $\mathcal{C}_\mathbb{R}$ where all x^i are real, $h = 0$, since it's purely imaginary. Then we have three classes of critical points: those with $h > 0$, $h = 0$ and $h < 0$. Since the n_i are determined by upward flow, critical points with $h > 0$ cannot have upward flows intersecting with $\mathcal{C}_\mathbb{R}$. Critical points with $h = 0$ can only have the trivial upward flow intersecting with $\mathcal{C}_\mathbb{R}$, hence for them $n_i = 0$. Lastly, for $h < 0$ the n_i are unconstrained. Hence we find that in this special case

$$\mathcal{C}_\mathbb{R} = \sum_{h=0} \mathcal{C}_{p_i} + \sum_{h<0} n_i \mathcal{C}_{p_i}. \tag{4.2.14}$$

The Airy function revisited

Let us now go back briefly to the Airy function. Its two critical points at $z = \pm 1$ have Morse index 1, and the value of h there were given in (4.2.6). If λ is not purely imaginary, there is no Morse flow between $x = \pm 1$, however if λ is purely imaginary, as we mentioned before, there is a flow line connecting $z = \pm 1$. Moreover, since h is constant on the real axis, the upward and downward flows can only intersect at $z = \pm 1$. For λ that are not purely imaginary, it follows that the downward flows from $z = \pm 1$ generate what we called $\mathcal{C}_2, \mathcal{C}_3$, which are exactly the generators of the relative homology $H_1(\mathbb{C}, \mathbb{C}_{<T})$ of integration cycles that we found by more primitive means. The exotic integration cycle equals $\mathcal{C} = n_{+1}\mathcal{C}_{+1} + n_{-1}\mathcal{C}_{-1}$. As long as Re $\lambda \neq 0$, there is no flow between $z = \pm 1$ and it follows by choosing the orientation as in the picture that $n_\pm = 1$. So $\mathcal{C} = \mathcal{C}_{-1} + \mathcal{C}_{+1}$, as expressed in (4.2.5).

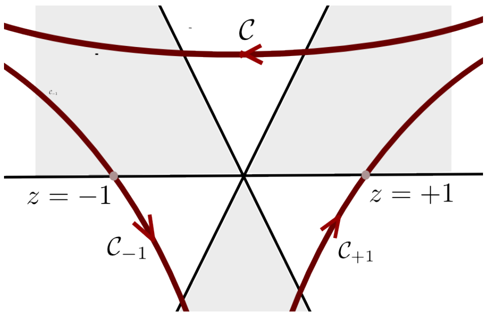


Figure 5: Lefschetz thimbles for the Airy function.

Degeneracy of critical points: critical subsets

The Morse functions that we will use will in general not have isolated critical *points*, but rather its critical subsets will form a (collection of) submanifolds, since in general there may be directions in which h has zero derivative; h is constant on such subsets. This requires a generalization of the techniques above. The class of functions whose critical subsets form submanifolds of M, but still have the same properties in the transverse directions to the critical subsets, consists of *Morse-Bott* functions. The Morse index in this case is usually denoted by $(i_-, i_+$, where i_- is the dimension of the unstable manifold and $i_+ = \dim M - i_-$ is the dimension of the stable manifold and the critical subset. We will assume that all functions used hereafter are of this type, that is, we can apply Morse-Bott theory to them.

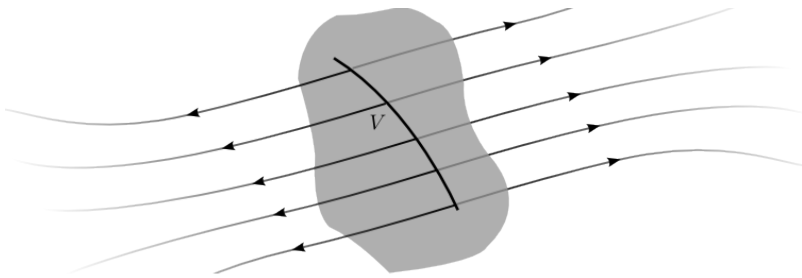


Figure 6: Downward flow from a middle-dimensional subset V of the critical set.

After complexification, all dimensions double, so we need to count dimensions to determine how many extra boundary conditions we should impose on the flow to get cycles $\mathcal{C}_p$ of middle dimension. Suppose we have a Morse-Bott function h, and let $N^{(2r)}$ be a connected component of the collection of critical subsets of h, of real dimension $2r$. There are $2n - 2r$ real dimensions normal to N and we assume that h is non-degenerate in those directions. Then there are $2n - 2r$ nonzero eigenvalues of the Hessian of h of which half are negative. Using the flow argument again, the values at $s = 0$ of flows that start at N form a submanifold $\mathcal{N}$ of real dimension $2r + (n - r) = n + r$. To get a middle-dimensional cycle, we need to impose an extra r conditions: we need to *choose some r-dimensional cycle $V^{(r)} \subset N^{(2r)}$ on which flows should start*. We see that h is of index $(2r, 2n - 2r)$.

4.3 Morse theory on infinite-dimensional M

There are a few generalizations that will be important to us. Firstly, one can apply Morse theory on a complex manifold, which is called *Picard-Lefschetz theory*. This shall be discussed in section 4.2. Furthermore, we shall need the infinite-dimensional version of Morse theory in chapters 6 and 8, where M

becomes infinite-dimensional. The infinite-dimensional version of Morse theory is called *Floer theory* and is technically more involved. However, the conceptual ideas of Morse theory still generalize, as long as the flow equations are elliptic.

Recall that on $\mathbb{R}^n$ a differential operator can be denoted as $D = \sum_{\alpha,|\alpha|=0}^{n} b_\alpha(x)\partial^\alpha$, where $x \in \mathbb{R}^n$ and α is a multi-index. D is elliptic if its characteristic polynomial (or symbol) $\sigma(x) \sim \sum_{|\alpha|=n} b_\alpha(x)p^\alpha$ is non-zero for $p^\alpha \neq 0$. This notion can be easily generalized to differential operators between fibers, where the appropriate generalization is that $\sigma(x)$ is invertible away from 0. An elliptic differential operator D retains most of the desirable properties of differential operators on finite-dimensional spaces: for instance, by the Fredholm alternative the kernel (the solution space of $Df = 0$) of D and its elliptic adjoint D^* is finite-dimensional and their solutions are well-behaved by regularity theorems. Moreover, it allows to rigorously define a relative Morse index, called the Conley index. To do this, analytical elliptic estimates are central in showing that these are well-behaved; we shall take as a mathematical fact that ellipticity validates the use of Morse theory in the infinite setting: more mathematical background can be found, for instance, in [13].

4.4 Exotic integration cycles and localization

The distinguishing feature of using exotic integration cycles is that one can formulate the path integrals on an exotic cycle in terms of open σ-models. In this story, the boundary $\partial\Sigma$ plays an essential role. In this section, we want to make precise the subtleties that appear at the boundary and to view the new duality from a different angle: we give an alternative explanation how we can localize the 1-dimensional σ-model on an exotic integration cycle for 0-dimensional quantum mechanics. More explicitly, we want to describe the Poincaré dual $\Psi_{top} = \eta[\mathcal{C}]$ to the exotic integration cycle $\mathcal{C}$ such that

$$Z = \int_{\mathcal{C}\subset M_{\mathbb{C}}} \Omega \exp S = \int_{M_{\mathbb{C}}} \Omega \exp S \wedge \Psi_{top} = \int_{M_{\mathbb{C}}} \Omega \exp S \wedge \eta_{\mathcal{C}}. \tag{4.4.1}$$

The open Landau-Ginzburg model

So we want to look at exotic integration cycles from the viewpoint of $\mathcal{N} = 1$ supersymmetric quantum mechanics, for this we choose its worldline to be $L = (-\infty, 0] = \mathbb{R}_-$ and its target space $M_{\mathbb{C}}$ the complexified phase space of 0-dimensional quantum mechanics. We recall its action (C.2.13)

$$S_{top} = \int ds \left(\frac{\partial\phi^i}{\partial s} + g^{ij}\frac{\partial h}{\partial\phi^j}\right)^2 = S_{phys} + \int_L dh = S_{phys} + \left(h(\phi(0)) - h(\phi(-\infty))\right). \tag{4.4.2}$$

We first consider an isolated non-degenerate critical point p of the Morse function h. Then we are interested in the path integral with the boundary condition $\phi^i(s) \to p$ as $s \longrightarrow -\infty$. That is, we want to consider the path integral

$$\Psi_{top}(p) = \int_{\phi(-\infty)\in p, \phi(0)=\phi_0} \mathcal{D}\phi(s)\mathcal{D}\psi(s)\mathcal{D}\chi(s) \exp\left(i\lambda S_{top}(\phi,\psi,\chi)\right). \tag{4.4.3}$$

Here we do not have any operator insertions, as they do not change the concepts discussed here. From section A.4 we know that this path integral calculates a state $\Psi_{top}(p)$ in the Hilbert space associated to the boundary at $s = 0$. From (4.4.2) we see that $\Psi_{phys}(p) = \exp\left(-\lambda(h(0) - h(-\infty))\right)\Psi_{top}(p)$. We assumed there are no interpolating flow lines: this means that $\Psi_{phys}(p)$ will be Q-invariant: $Q\Psi_{phys}(p) = e^h de^{-h}\Psi_{phys}(p) = 0$. Hence $d\Psi_{top}(p) = 0$, both statements are just a consequence of our identification of Q with the de Rham differential and (C.2.16).*

The fact that the theory localizes on solutions of the flow equation (4.2.7) (with $w^i = \phi^i$) means that $\Psi_{top}(p)$ gives a δ-function with support on the set $\mathcal{C}_p$ of all points that are reached by downward flow

*Suppose that there was a downward flow between two points with Morse index p and $p-1$, whose classical ground states we denote by $|p\rangle, |p-1\rangle$. Then $\partial|p\rangle = |p-1\rangle \neq 0$ and $d|p-1\rangle = |p\rangle + \ldots$. $\Psi_{top}(p)$ calculates the Poincaré dual of a family of flow lines, that contains a subfamily that interpolates between $|p\rangle$ and $|p-1\rangle$. We interpreted $\Psi_{top}(p)$ as a state at $s = 0$, which for this subfamily contains exactly the point of Morse index $p-1$. Hence we see that $d\Psi_{top}(p) \neq 0$.

from p. Note that any flow line is uniquely fixed by specifying one point (say at $s = 0$) on the flow line, by uniqueness of flows. Our goal is to use $\Psi_{top}(p)$ to define a path integral that is restricted to integrate over $\mathcal{C}_p$.

We now assume that $\mathcal{C}_p$ is a middle-dimensional cycle in the complex manifold M. This middle-dimensional cycle is an n real-dimensional complex submanifold, hence it is the zero set of n anti-holomorphic functions, say $x^{\bar{i}} = 0, i = 1 \ldots n$. This means that the state at $s = 0$ calculated by the path integral is the Poincaré dual to $\mathcal{C}_p$:

$$\Psi_{top}(p) = \delta(x^{\bar{1}}) \ldots \delta(x^{\bar{n}}) d\phi^{\bar{1}} \ldots d\phi^{\bar{n}} = \delta(x^{\bar{1}}) \ldots \delta(x^{\bar{n}}) \psi^{\bar{1}} \ldots \psi^{\bar{n}} = \delta(x^{\bar{1}}) \ldots \delta(x^{\bar{n}}) \delta(\psi^{\bar{1}}) \ldots \delta(\psi^{\bar{n}}), \tag{4.4.4}$$

which is what we were looking for. The last equality follows since a δ-function for a fermionic variable ψ is equivalent to the variable itself since $\int d\psi \delta(\psi) = \int d\psi \psi = 1$. We now recall the logic presented at the end of section (A.4): to obtain a number we need to pair $\Psi_{top}(p)$ with a dual state $\tilde{\Psi}$ and integrate over M: the number calculated is the value of the quantum mechanical path integral (6.2.2) evaluated on an exotic integration cycle $\mathcal{C} = \mathcal{C}_p$. $\tilde{\Psi}$ must be given by a form Υ inserted at $s = 0$. If we assume that $M_{\mathbb{C}}$ is Calabi-Yau[§], then $M_{\mathbb{C}}$ has a non-vanishing top holomorphic form Ω. Hence an appropriate choice for Υ is

$$\Upsilon = \Omega \exp S = \Omega_{i_1 \ldots i_n}(\phi^i) d\phi^{i_1} \wedge \ldots \wedge d\phi^{i_n} \exp S|_{s=0} = \Omega_{i_1 \ldots i_n}(\phi^i) \psi^{i_1} \ldots \psi^{i_n} \exp S|_{s=0},$$

where $\mathcal{S}$ is the complexified action of the dual theory, a holomorphic function on M. This means

$$Z = \int_{M_{\mathbb{C}}} \Psi_{top}(p) \wedge \Upsilon = \int_{\mathcal{C}_p} \Upsilon = \int_{\mathcal{C}_p} \Omega \exp S. \tag{4.4.5}$$

This final compact answer is just a generalization of, for instance, the path integral (4.2.5). As before the Morse techniques make this a formally convergent expression. Since $\Psi_{phys}(p)$ was Q-invariant, it is straightforward to check Q-invariance of the path integral Z.[*] The boundary conditions on the fermions follow straightforwardly from varying the action and forcing the boundary contributions to vanish at $s = 0$, the bulk variations are killed by the equations of motion. This gives

$$\delta I_f = \delta \int_{(-\infty,0]} ds \left(\psi_i D_s \chi^i + \psi_{\bar{i}} D_s \chi^{\bar{i}} \right) = \ldots + \psi_{\bar{i}} \delta \chi^{\bar{i}}|_{s=0}. \tag{4.4.6}$$

Note that the fermionic delta functions set the holomorphic parts $\psi^i = \psi^{(1,0)}$ to zero at $s = 0$, so the $(0,1)$ part $\chi^{\bar{i}}$ has to vanish at the boundary at $s = 0$, since we only had a constraint on the $(1,0)$-part of ψ at $s = 0$.

[§]Note that this is not too constraining, as we saw in the previous sections that in most cases, the duality works if $M_{\mathbb{C}}$ is (almost) hyperkähler. Since every hyperkähler manifold is Calabi-Yau, our construction here will 'generically' be applicable.

[*]S_{top} is Q invariant since by construction S_{top} is Q-exact. The fermions are Q-invariant by construction, see (??). Furthermore by holomorphicity of $\mathcal{S}$, $\mathcal{S}$ and Ω are only a function of ϕ^i: the supersymmetry transformation $\{Q, \phi^i\} = \psi^i$ give squares of fermions upon variation of Z, hence vanishes also.

5

Exotic integration cycles and gauge symmetry

In this section we extend the discussion of the previous chapter to theories with gauge symmetry. The gauge redundancy introduces subtleties in the classification of critical 'points', since they are gauge orbits of the symmetry. Moreover, the decomposition of a given cycle in terms of Lefschetz thimbles has to be suitably extended. Having discussed these issues, we will then see how this pairs with localization in the 1-dimensional gauged open σ-model, which is dual to gauged quantum mechanics. The generalization to infinite manifolds of this model will be used in chapter 8.

5.1 Gauge-invariant exotic integration cycles

Suppose M is a manifold with some G-action on it. We shall assume G is compact. We will denote their complexifications by $M_{\mathbb{C}}$ and $G_{\mathbb{C}}$ respectively. Again our goal is to use the real part of a holomorphic function as a Morse function h, whose critical subsets form $G_{\mathbb{C}}$-orbits, to find an exotic integration cycle for the integral

$$Z = \int_{\mathcal{C}=M} d^n x \exp\left(i\lambda f(x^1,\dots,x^n)\right), \tag{5.1.1}$$

where f is a polynomial. This will be complexified to

$$\mathcal{Z} = \int_{\mathcal{C}'\subset M_{\mathbb{C}}} dz \exp \mathcal{S} \tag{5.1.2}$$

where $\mathcal{S}$ now becomes a holomorphic function of the z^i, and our Morse function will be $h = \text{Re}\,\mathcal{S}$. We denote critical orbits by $\mathcal{O}^G$ and $\mathcal{O}^{G_{\mathbb{C}}}$ respectively.

Free action

The simplest case is when G acts freely on M: $G_{\mathbb{C}}$ then also acts freely on $M_{\mathbb{C}}$, and the quotients M/G and $M_{\mathbb{C}}/G_{\mathbb{C}}$ are non-singular manifolds again. In that case, the path integral Z can be written as

$$\mathcal{Z} = \int_{M/G} dx' \exp \mathcal{S}, \tag{5.1.3}$$

where the measure dx' can be obtained for instance by integrating over the fibers of $M \longrightarrow M/G$. In this case, critical orbits of h on M correspond to honest critical points on M/G, to which we can apply the techniques developed in section 4.2 again. So how should be interpret this on $M_{\mathbb{C}}$? Any critical orbit $\mathcal{O}^{G_{\mathbb{C}}}$ is a copy of $G_{\mathbb{C}}$, since $G_{\mathbb{C}}$ acted freely on $M_{\mathbb{C}}$, and $G_{\mathbb{C}}$ is isomorphic to the cotangent bundle T^*G.* An example is $G_{\mathbb{C}} = GL(n,\mathbb{C})$, whose maximal compact subgroup is $U(n)$. The Lie algebra $\mathfrak{u}(n)$ consists of antihermitian matrices. $\mathfrak{g}^-$ is then given by all hermitian matrices.

Now the middle-dimensional homology of $\mathcal{O}^{G_{\mathbb{C}}}$ has rank 1, generated by the zero section of T^*G, which is just $G \subset T^*G$. So if G acts freely on M, every critical orbit will contribute one classical ground state; in more mathematical terms they contribute one generator to the middle-dimensional relative homology of $M_{\mathbb{C}}$, analogous to the non-gauged case.

*This follows from the theory of *symmetric spaces*. If $G_{\mathbb{C}}$ is the complexification of a compact semisimple Lie group G, there is an unique involution, the *Cartan involution* i, that leaves the maximal compact subgroup K of $G_{\mathbb{C}}$ fixed. This involution also acts on Lie$(G_{\mathbb{C}})$, which splits into the two eigenspaces $\mathfrak{g}^{\pm}$ associated to the ±1-eigenvalue of ι. By the polar decomposition of $G_{\mathbb{C}}$, we then have $\mathfrak{g}^- \times K \cong G_{\mathbb{C}}$ through the diffeomorphism $(X,k) \longmapsto k \exp X \in G_{\mathbb{C}}$. It turns out that $\mathfrak{g}^- = \mathfrak{k}^*$, which shows $T^*G = G_{\mathbb{C}}$.

Non-free action

The more subtle case is when G does not act freely on M: the G-action may have some fixed points on M. In this case we cannot apply our previously developed techniques to M/G, since as a manifold it is singular. So we cannot apply our Morse theory techniques there: we need to stay on M and $M_{\mathbb{C}}$; we assume that h has a finite number of critical orbits on M and $M_{\mathbb{C}}$.* A critical orbit $\mathcal{O}^G$ in M has a complexification $\mathcal{O}^{G_{\mathbb{C}}}$ that lies in $M_{\mathbb{C}}$. Topologically, $\mathcal{O}^{G_{\mathbb{C}}}$ is isomorphic to $T^*\mathcal{O}^G$, hence such an orbit will again contribute one generator to the relative homology of $M_{\mathbb{C}}$.

However, we might have critical orbits in $M_{\mathbb{C}}$ that do not arise as a complexification from a critical orbit in M. We want to argue that only critical orbits which are *semistable* matter. Semistable orbits are critical orbits that admit a point where the moment map μ_G for the G-action vanishes, $\mu_G = 0$. If G acts freely on the orbit or the stabilizer is at most a finite subgroup, we call it stable. If a critical orbit is not semistable, we call it *unstable*¶.

To write down the Morse flow equation, we need a metric on $M_{\mathbb{C}}$. Generically we can only pick a G-invariant Kähler metric $g_{i\bar{j}}$ whose Kähler form ω is odd under complex conjugation and such that the G-action preserves ω. This gives a moment map μ_G for the G-action on $M_{\mathbb{C}}$ (see appendix A.2), whose defining equation is

$$d\mu_{G,V} = \iota_V \omega, \qquad V \in \mathfrak{g}. \tag{5.1.4}$$

The pointe is that μ is conserved along Morse flows: dropping the subscript G we just compute

$$\begin{aligned}\frac{d\mu_V}{ds} &= \frac{\partial \mu_V}{\partial z^i}\frac{dz^i}{ds} + \frac{\partial \mu_V}{\partial z^{\bar{i}}}\frac{dz^{\bar{i}}}{ds} = -V^{\bar{j}}\omega_{\bar{j}i}g^{i\bar{k}}\frac{\partial h}{\partial z^{\bar{k}}} - V^j\omega_{j\bar{i}}g^{\bar{i}k}\frac{\partial h}{\partial z^k} = -V^{\bar{j}}\frac{\partial(i\mathrm{Im}\,\mathcal{S})}{\partial z^{\bar{j}}} - V^j\frac{\partial(-i\mathrm{Im}\,\mathcal{S})}{\partial z^j} \\ &= -\iota_V d\mathrm{Im}\,\mathcal{S} = 0\end{aligned} \tag{5.1.5}$$

Here we subsequently used the defining equation for the moment map in index notation $\frac{\partial \mu_V}{\partial z^i} = V^{\bar{j}}\omega_{\bar{j}i}$, the Cauchy-Riemann equations for the holomorphic function $\mathcal{S}$, which tells us that for $z^i = x^i + iy^i$

$$\frac{\partial \mathrm{Re}\,\mathcal{S}}{\partial z^i} = \frac{1}{2}\left(\frac{\partial \mathrm{Re}\,\mathcal{S}}{\partial x^i} - i\frac{\partial \mathrm{Re}\,\mathcal{S}}{\partial y^i}\right) = \frac{1}{2}\left(\frac{\partial \mathrm{Im}\,\mathcal{S}}{\partial y^i} + i\frac{\partial \mathrm{Im}\,\mathcal{S}}{\partial x^i}\right) = \frac{1}{2}\left(-i\frac{\partial i\mathrm{Im}\,\mathcal{S}}{\partial y^i} + \frac{\partial i\mathrm{Im}\,\mathcal{S}}{\partial x^i}\right) = \frac{\partial i\mathrm{Im}\,\mathcal{S}}{\partial z^i}$$

and likewise that

$$\frac{\partial \mathrm{Re}\,\mathcal{S}}{\partial z^{\bar{i}}} = \frac{\partial(-i\mathrm{Im}\,\mathcal{S})}{\partial z^{\bar{i}}},$$

the Morse flow equation in local complex coordinates $(z^i, z^{\bar{i}})$ and the fact that $\mathcal{S}$ is G-invariant $\mathcal{L}_V\mathcal{S} = (d\iota_V + \iota_V d)\mathcal{S} = 0$, so $\mathrm{Im}\,\mathcal{S} = \frac{1}{2i}\left(\mathcal{S} - \overline{\mathcal{S}}\right)$ is too. Note that $\iota_V\mathcal{S} = 0$.

Since μ vanishes on M, any critical $G_{\mathbb{C}}$-orbit $\mathcal{O}^{G_{\mathbb{C}}}$ that is connected to M by a Morse flow is semistable. Now our original integration cycle $\mathcal{C} = M$ for the path integral was certainly semistable, so our exotic integration cycle will have to consist of semistable cycles too. To see this, we consider for real λ the decomposition $\mathcal{C}' = \sum_\sigma n_\sigma \mathcal{C}_\sigma$. The critical points labeled by σ fall into three categories: either $h = 0$, $h < 0$ or $h > 0$ at σ. Only $h \leq 0$ points contribute, since only those can have upward flows that intersect $\mathcal{C}$ where $h|_{z^i = x^i} = \mathrm{Re}\left(i\lambda f(x^i)\right) = 0$, since h is strictly increasing along upward flows. Therefore, if $\langle \mathcal{C}, \mathcal{K}_\sigma\rangle \neq 0$, the critical orbits σ must have a subset where $\mu = 0$, which follows automatically by

*Note that the number of critical orbits does not have to be the same on M and $M_{\mathbb{C}}$ (for instance, $i\lambda\left(x^3 + x\right)$ has no critical points on the real line, but has two in the complex plane).

¶An example of this is the following: consider $G_{\mathbb{C}} = SL(2,\mathbb{C})$ which is the complexification of $G = SU(2)$. An unstable orbit of $G_{\mathbb{C}}$ is $\mathbb{C}P^1$, which is a homogeneous $SU(2)$-space. A nonempty topological space X is a homogeneous G-space if for every $x, y \in X$ there is a $g \in G$ such that $g \cdot x = y$). This can be seen as follows: $SU(2) \cong SO(3)$ which acts transitively on S^2 by rotations. But S^2 as the Riemann sphere is diffeomorphic to $\mathbb{C}P^1$. So $SU(2)$ acts transitively on $\mathbb{C}P^1$. But then G has to act trivially on $\mathbb{C}P^1$, since if $\mu = 0$ somewhere, it has to vanish identically on $\mathbb{C}P^1$ since the equation $\mu = 0$ is G-invariant. This implies that $V = 0$ for all vector fields V associated to generators of G by the defining equation for the moment map. We see that such an orbit cannot be semistable.

conservation of μ along flows and the fact that $\mu = 0$ on $\mathcal{C}$. This follows from its defining equation (??): under complex conjugation, the symplectic form is odd, whereas V is even*. Therefore, μ must be odd under complex conjugation and it has to be 0 on $\mathcal{C} = M$ (since μ is defined by its derivative, we can always fix this constant at will). Hence all contributing critical points are semistable. A more explicit indication that semistability is required follows from equation (5.3.1): the condition for a critical point of h in that situation is that $\mu = 0$ at any critical point.

Because $\mathcal{O}$ is preserved by G, it will be isomorphic to G/P, where $P \subset G$ is the stabilizer of the G-action: P measures how many fixed points G has on $\mathcal{O}$. If P is at most finite we will call $\mathcal{O}$ *stable*. The reason for this terminology is that the stabilizer P is a discrete subgroup of G and as a consequence there exists a smooth covering map $\pi : G \longrightarrow G/P$. This implies that G/P does not have singularities. In particular, the quotient $\mathcal{O}/G$ consists of just a point again, just as in the case where G acted freely on $\mathcal{O}$. For instance, $\pi : \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} \cong S^1$, where $\mathbb{Z}$ acts freely by addition on $\mathbb{R}$. The quotient $S^1/\mathbb{R}$ where $\mathbb{R}$ acts by translation on the circle is clearly a point.

Obtaining exotic integration cycles.

Let $\mathcal{O}_i^{G_\mathbb{C}}$ be a semistable critical orbit, where i is an index. It has a subspace $\mathcal{O}_i^G$ of points where $\mu_G = 0$. Then the Morse flow from $\mathcal{O}_i^G$ defines a cycle $\mathcal{C}_i$: it consists of all points that can be reached by downward flow from $\mathcal{O}_i^G$; alternatively, it consists of the collection of values at $s = 0$ of all possible downward flows from $\mathcal{O}_i^G$. Then our exotic integration cycle $\mathcal{C}'$ will usually be a subset of $\mu_G^{-1}(0)$, and always be a linear combination

$$\mathcal{C}' = \sum_i n_i \mathcal{C}_i. \tag{5.1.6}$$

Earlier, we determined the coefficients n_i by taking intersection products of $\mathcal{C}$ with upward flows, because their relative homology was the natural dual to the relative homology of downward flows. However, in this case, this does not quite work, because upward and downward flows intersect in an entire orbit $\mathcal{O}_i^G$: the intersection number would equal the Euler characteristic of this orbit, which vanishes for a stable orbit.

This follows from regularization of self-intersection numbers. To compute the intersection number $X \cap X$ for a manifold X, we deform the second factor to X' using some vector field normal to X. This vector field will have some zeroes in general, so $X \cap X'$ consists of a finite amount of points, corresponding to the finite number of zeroes of our chosen vector field. As we pointed out in the beginning of this chapter, the graded sum of zeroes of any smooth vector field on X computes exactly the Euler number of X, which we then formally call the self-intersection number of X.

Since H is a compact connected semisimple Lie group, its Euler number vanishes. This follows easily from the fact that on every Lie group there always exists a global non-zero vector field: just take any vector $V \in \mathfrak{g}$, which under left multiplication $dL_a, a \in G$ can be extended to a global vector field on G, which is non-vanishing by the group properties of G. Since the Euler characteristic counts with signs the zeroes of all smooth vector fields by the Poincaré-Hopf theorem, it follows easily that $\chi(G) = 0$ for any compact connected G.

So to get a nontrivial answer, the appropriate thing to do is to use the dual to the orbit $\mathcal{O}_i^G$, which is any fiber $\mathcal{O}_i'^G$ in $T^*\mathcal{O}_i^G$. The upward flow $\mathcal{K}_i$ from $\mathcal{O}_i'^G$ then intersects $\mathcal{C}_i$ only in the base-point p_i of $\mathcal{O}_i'^G$. In this case we again find $n_i = \langle \mathcal{C}, \mathcal{K}_i \rangle$.

5.2 The 1-dimensional gauged open σ-model

In the previous section, we discussed supersymmetric quantum mechanics in the presence of a superpotential. At this point, we generalize that model by adding a gauged symmetry of the target space. We

*By Darboux's lemma, the symplectic form locally always is of the form $\omega = -i \sum_i dz^i \wedge dz^{\bar{i}}$, which is clearly odd under $z^i \leftrightarrow z^{\bar{i}}$, whereas $V = V^i \frac{\partial}{\partial z^i} + V^{\bar{i}} \frac{\partial}{\partial z^{\bar{i}}}$ is even.

take the target space M to be Kähler, with real dimension $2n$ and Kähler form ω. Recall that this gives us $\mathcal{N} = 2$ worldsheet supersymmetry. We assume M has a Lie group of symmetries G that acts on M and gauge this symmetry. The group action on M is given by a homomorphism $\pi : G \to \text{Diff}(M)$ and has associated Killing vectors $\{\xi_a^i\}$.* Hence to any $y \in \mathfrak{g}$ we can associate a Killing vector field

$$\xi^i(y) = y^a \xi_a^i. \tag{5.2.1}$$

The moment map of the G-action is denoted by μ. Our goal is to describe the 1d gauged Landau-Ginzburg model, which governs the map $\Phi : L \longrightarrow M$, coupled to vector multiplets for the G-action.

One way to construct this model is by dimensional reduction from the $\mathcal{N} = 1$ 4d gauged σ-model, with $\Phi : \mathbb{R}^4 \longrightarrow M$. The idea is to first reduce to a 2-dimensional worldsheet, then to a 1-dimensional worldline. In doing the dimensional reduction, we will do an additional topological twist, as we will need a twisted version of this model with $\Phi : \mathbb{R}_+ \times W \to M$ in chapter 8.

The 4-dimensional field content is contained in $\dim M$ chiral multiplets and a vector multiplet. We choose coordinates $y^\mu, \mu = 0, \ldots 3$ on $\mathbb{R}^4$. From the chiral multiplets $(\phi^i, \psi^{i\alpha})$, the σ-model map ϕ has components $\phi^i, i = 1, \ldots, \dim_{\mathbb{C}} M$, which are local coordinates on M, and the $\psi^{i\alpha}$ are Weyl spinors. In the vector multiplet $(A_\mu, \chi^\alpha, \overline{\chi}^{\dot\alpha})$, the bosonic field is the gauge field $A = A_\mu dy^\mu$ and we have a Weyl spinor χ^α with its conjugate. We shall take all gauge fields in the adjoint representation.

Upon dimensional reduction to a 2-dimensional worldsheet A_0, A_1 become a gauge field in two dimensions, and A_2, A_3 become $\mathfrak{g}$-valued scalars. We define the $\mathfrak{g}$-valued scalar $\sigma = A_2 - iA_3$ and its complex conjugate $\overline{\sigma} = A_2 + iA_3$ To let the model localize on the Morse flow equation, we need to do an A-twist, using the topological A-model supercharge $Q_A = \overline{Q}_+ + Q_-$ discussed in chapter 3. In order to do the A-twist, we assume the superpotential W is quasi-homogeneous: the scalar fields should admit a $U(1)$ symmetry that transforms $W \to e^{i\alpha} W$; this will be detailed further in section (6.3). Q cannot generate translations, as translation have M_A-eigenvalue ± 1. However, it turns out that in this case Q is not nilpotent: rather we have $Q^2 = [\sigma, .]$. Hence, only on gauge-invariant fields and states, Q is still nilpotent: on such states, we can define the cohomology of Q.

The lengthy details of this construction can be found in [10, 11], which we will not repeat in full here. From our earlier constructions, the results that follow below are exactly that one would expect.

Localization

Localization implies that the path integral localizes to fixed points of the fermionic supersymmetry variations. For the fermions in the vector multiplet, one finds that this implies that

$$*F + \mu = 0, \qquad D_\mu \sigma = \xi(\sigma) = [\sigma, \overline{\sigma}] = 0, \tag{5.2.2}$$

where $F = dA + A \wedge A$ is the curvature of A, while μ is the moment map for the G-action. For the fermions in the chiral multiplets, this means

$$\overline{\partial}_A \phi^i + g^{i\overline{j}} \frac{\partial \overline{W}}{\partial \overline{\phi^j}} = 0, \qquad \partial_A \phi^{\overline{i}} + g^{\overline{i}j} \frac{\partial W}{\partial \phi^j} = 0. \tag{5.2.3}$$

The second set of equations in (5.2.2) generically imply that $\sigma = 0$, so we will assume this from now on. Of the remaining the last two equations are familiar: they are the perturbed equations for a (anti)holomorphic map, further perturbed by the presence of the gauge field in the covariant derivative. The perturbation spoils the 2-dimensional symmetry of the flow equations: we cannot interpret them as ordinary Cauchy-Riemann equations anymore. This issue will be explored further in section 6.2.

*Recall that any element in $\mathfrak{g}$ can be decomposed with respect to a basis $\{\xi_a\}$ of $\mathfrak{g}$, for any $y \in \mathfrak{g}$ we write $y = y^a \xi_a$. Now by exponentiating, we get for any basis element of $\mathfrak{g}$ a vector $\tilde{\xi}_a = \frac{d}{dt}\Big|_{t=0} \pi(\exp ty)$ in TM. For convenience we drop the $\sim$ and denote this vector as ξ_a^i, which is a Killing vector. We'll assume the G-action is effective, so the space of Killing vectors is isomorphic to $\mathfrak{g}$.

Reduction to 1 dimension

For the purposes outlined in chapter 8, we want to discuss a further dimensional reduction to 1 dimension, upon which we can interpret (5.2.3) and (5.2.2) as flow equations in the $y^0 \equiv s$ direction. We then end up in the situation familiar from section 4.4 with a 1-dimensional worldline.[†] The gauge field is broken up further to a 1-dimensional gauge field A_0 and an adjoint-valued real scalar field A_1; in the gauge $A_0 = 0$ the flow equations in (5.2.3) and (5.2.2)) reduce to

$$\frac{dA_1}{ds} = -\mu = -\frac{\partial h}{\partial A_1}, \tag{5.2.4}$$

$$\frac{d\phi^i}{ds} = iA_1^a\xi_a^i - g^{i\bar{j}}\frac{\partial\overline{W}}{\partial\phi^{\bar{j}}} = iA_1^a\omega^{i\bar{j}}\frac{\partial\mu_a}{\partial\phi^{\bar{j}}} - g^{i\bar{j}}\frac{\partial(W+\overline{W})}{\partial\phi^{\bar{j}}} = -g^{i\bar{j}}\frac{\partial(A_1^a\mu_a + 2\text{Re }W)}{\partial\phi^{\bar{j}}} = -g^{i\bar{j}}\frac{\partial h}{\partial\phi^{\bar{j}}} \tag{5.2.5}$$

where we see[*] that we can identify the appropriate Morse function

$$h = A_1^a\mu_a + 2\text{Re }W. \tag{5.2.6}$$

So equations (5.2.4) and (5.2.5) define Morse flow on $M \times \mathfrak{g}$.

Equivariant cohomology, observables and physical states

Recall that observables of supersymmetric quantum mechanics (chapter 7) and the A-model (chapter 3) sat in the cohomology of the topological supercharge Q, which corresponds to the de Rham cohomology of the target space M. In the gauged σ-model, the generalization of this idea is that observables sit in a cohomology compatible with the gauge symmetry: the equivariant cohomology of the target space. Moreover, we saw in chapter 7 that ground states of the theory also corresponded to elements in the target space cohomology; likewise here ground states will be elements in the target space equivariant cohomology. Some details of equivariant cohomology are briefly described in appendix A.1. As before, M is Kähler.

Consider $K = M \times \mathfrak{g}$. σ was a generator of the G-symmetry and so we can consider $\sigma \in \mathfrak{g}$ to be the generator of degree 2 of Sym$(\mathfrak{g}^\bullet)$, the space of symmetric polynomials. We can compute the action of $\mathcal{D}_0 = d + i_{X(\sigma)}$:

$$\left[\mathcal{D}_0, \phi^i\right] = \psi^i, \qquad \left\{\mathcal{D}_0, \psi^i\right\} = X^i(\sigma), \tag{5.2.7}$$

which are precisely the supersymmetry variations for the chiral multiplets. For the vector multiplet, one similarly finds that

$$[\mathcal{D}_0, A_\mu] = \lambda_\mu, \qquad \{\mathcal{D}_0, \lambda_\mu\} = -D_\mu\sigma, \qquad [\mathcal{D}_0, \sigma] = [Q, \sigma] = 0. \tag{5.2.8}$$

If this was the entire story, we would now use (A.1.12) as the equivariant cohomology. In our application, there are now two issues that require a slight generalization of (A.1.12): the complexification $G \to G_{\mathbb{C}}$ and the non-compactness of $M_{\mathbb{C}}$, which generally makes h unbounded from above and below.

The modification due to the complexification $G \to G_{\mathbb{C}}$ comes about since we also have the complex conjugate $\overline{\sigma}$. Hence, we should view σ and $\overline{\sigma}$ as coordinates on $\mathfrak{g}_{\mathbb{C}}$, regarded as a complex manifold. In doing so, we should replace

$$\text{Sym}(\mathfrak{g}^*) \Rightarrow \Omega^{0,\bullet}(\mathfrak{g}_{\mathbb{C}}), \tag{5.2.9}$$

the latter space consisting of $(0, j)$-forms on $\mathfrak{g}_{\mathbb{C}}$. Here $0 \leq j \leq \dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}}$. The grading comes by setting the grading of $\sigma, \overline{\sigma}, \eta$ to be $2, -2, -1$. Likewise, we should extend the twisted de Rham operator to

$$\mathcal{D} = \mathcal{D}_0 + \mathcal{D}_1, \qquad \mathcal{D}_1 = d\overline{\sigma}^a\frac{\partial}{\partial\overline{\sigma}^a} + [\sigma, \overline{\sigma}]^a\, i_{d\overline{\sigma}^a}. \tag{5.2.10}$$

[†] Note that we started from $\mathcal{N} = 1$, which means that we started out with 4 real supercharges (a 4-dimensional Weyl spinor has 2 complex components). After dimensional reduction to 2 dimensions and moreover twisting, we were left with 2 real topological (scalar) supercharges. Going to the 1-dimensional model, there are actually 4 real, scalar, supercharges again, since in 1 dimension a vector is the same as a scalar.

[*] Here we used the Kähler form: $g_{i\bar{j}} = -i\omega_{i\bar{j}}$ and the fact that the superpotential W is holomorphic.

This construction is compatible with the supersymmetry relations $[Q,\sigma]=0$, $[Q,\overline{\sigma}]=\eta$ and the identification $\eta=d\overline{\sigma}$: we get only antiholomorphic $(0,q)$-forms by repeatedly acting with Q. Now a calculation shows that $\mathcal{D}$ obeys $\mathcal{D}^2=\mathcal{L}_{X(\sigma)}$, so we can speak about the cohomology of $\mathcal{D}$ on the G-invariant subspace generated by σ, Ξ and its associated complex

$$\Xi=\left(\Omega^{0,\bullet}(\mathfrak{g}_{\mathbb{C}})\otimes\Omega^{\bullet}(M)\right)^{G},\qquad(\Xi,\mathcal{D}),\tag{5.2.11}$$

where $\Omega^{\bullet}(M)$ is the space of ordinary differential forms on M. The cohomology of $\mathcal{D}$ coincides with the cohomology of $\mathcal{D}_0$ on this space. To show this, one only needs to show that the action of $\mathcal{D}$ coincides with that of $\mathcal{D}_0$ on Ξ. To do so, it is sufficient to show that $\mathcal{D}\Psi=0$ implies that Ψ does not contain a factor of $d\overline{\sigma}$. A full proof of this can be found in [12].* This tells us that actually all the relevant equivariant forms in Ξ do not include $\overline{\sigma}$ and $d\overline{\sigma}$ at all! Hence an element of degree $2n+p$ in the cohomology of $(\Xi,\mathcal{D})$ generally looks like

$$\omega=\omega_{a_1\dots a_n k_1\dots k_p}\sigma^{a_1}\dots\sigma^{a_n}d\phi^{k_1}\wedge\dots\wedge d\phi^{k_p},\tag{5.2.12}$$

where a_i are indices on $\mathfrak{g}_{\mathbb{C}}$, k_i are indices on M, $0\le 2n\le\dim_{\mathbb{C}}\mathfrak{g}_{\mathbb{C}}$, $0\le p\le\dim_{\mathbb{R}}M$ and $0\le 2n+p\le\dim_{\mathbb{R}}M$.† The second issue when we deform the theory by a superpotential h, such that the supercharge is deformed to

$$\tilde{Q}=\exp(\lambda h)\mathcal{D}\exp(-\lambda h).\tag{5.2.13}$$

Localization by taking the limit $\lambda\longrightarrow\infty$ shows that the cohomology of classical ground states is given by the critical points of the Morse function h, see appendix 7. Physical fields sit in the cohomology of $\tilde{Q}$, which is equivalent to the cohomology of $\mathcal{D}$, which in turn is equivalent to the cohomology of $\mathcal{D}_0$, which is the equivariant cohomology of M.

Recall that h has to be bounded above to find an exotic integration cycle. Hence, we should restrict ourselves further to the space of differential forms on whose support h is bounded above, which we denote as $\Omega^{\bullet}_{h<\infty}(M)$. Hence to ensure that we can apply Morse theory, the twisted de Rham complex we should use is

$$\Omega^{\bullet}_{G,h<\infty}(M)=\left(\left(\Omega^{0,\bullet}_{h<\infty}(\mathfrak{g}_{\mathbb{C}})\otimes\Omega^{\bullet}_{h<\infty}(M)\right)^{G},\mathcal{D}\right).\tag{5.2.14}$$

The equivariant cohomology of M is the cohomology $H^{\bullet}_{G,h<\infty}(M)$ of this complex: an element is represented by an equivariant differential form given by (5.2.12), with the proviso that h is bounded on its support.

5.3 Gauge-invariant critical orbits

Using (5.2.6), the conditions for a critical point of h become

$$\frac{\partial h}{\partial\phi^i}=\omega_{\bar{j}i}\xi^{\bar{j}}(A_1)-\frac{\partial W}{\partial\phi^i},\qquad\frac{\partial h}{\partial A_1}=\mu\implies\frac{\partial W}{\partial\phi^i}=0,\qquad\mu=0.\tag{5.3.1}$$

This follows by multiplying the first equation by $\xi^i(A_1)$, and using $\xi^i(A_1)\frac{\partial W}{\partial\phi^i}=0$ since W is G-invariant, we find that $\xi^i(A_1)\omega_{\bar{j}i}\xi^{\bar{j}}(A_1)=0$, so $\xi(A_1)=0$. These describe exactly a semistable critical orbit of h: gauge invariant critical orbits of W on which the moment map μ vanishes. W is gauge-invariant and holomorphic by construction (by definition of the superpotential), so it is invariant under the $G_{\mathbb{C}}$-action. Hence, critical points of W are $G_{\mathbb{C}}$-orbits. We assume there are finitely many critical $G_{\mathbb{C}}$-orbits, which are non-degenerate.*

*The idea is that given a form β with $\mathcal{D}\beta=0$ containing $d\overline{\sigma}$, one can always lower the degree in $d\overline{\sigma}$ by shifting the form by $\mathcal{D}\alpha$ for some α: this follows from the term $d\overline{\sigma}\frac{\partial}{\partial\overline{\sigma}}$ in (5.2.10). Note that $\mathcal{D}$ automatically eliminates a top $d\overline{\sigma}$ form. By recursion, one concludes that β must be independent of $\overline{\sigma}$ and $d\overline{\sigma}$.

†So far, we actually have used only half of all the fermions in our discussion. The field A_0 is set to zero, but acts as a Lagrange multiplier for the constraint that states should be G-invariant. Since the other half of the fermions are all complex conjugate to the ones discussed here, the discussion for them is analogous. However, to get the right $\mathcal{D}$-variations of those fermions, one has to deform $\mathcal{D}$ just as in (5.2.13). This subtle detail is discussed in more detail in [12], but is inconsequential for us here.

*Note that this what one would expect: vanishing of the scalar potential signals unbroken supersymmetry, while vanishing of the moment map signals G-invariance. Now each critical orbit will contribute to the equivariant cohomology of $M\times\mathfrak{g}$.

Only semistable $G_{\mathbb{C}}$-orbits contribute to the equivariant cohomology, because only there h has a true extremum. If G acts freely on M, the equivariant cohomology of $\mathcal{O}$ is the ordinary cohomology of $\mathcal{O}/G$, which is just a point. Hence a stable orbit contributes a 1-dimensional element to the equivariant cohomology of $M \times \mathfrak{g}$, whose degree equals the Morse index of h at $\mathcal{O}$, which is always $\frac{1}{2}\dim M$.*

This also holds if G has a finite stabilizer P. If P is not finite, $\mathcal{O} \cong G/P$, then the equivariant cohomology of $\mathcal{O}$ equals that of P acting on a point: it is the cohomology of $\mathcal{D}_0$ acting on the P-invariant part of Sym$(\mathfrak{p}^\bullet)$. Its contribution to the equivariant cohomology of $M \times \mathfrak{g}$ consists of elements of degree $\frac{1}{2}\dim M$ and generically infinitely many classes of higher degree.

Flat directions at critical orbits

Supersymmetric ground states correspond to minima of the scalar potential, which for the 1d gauged Landau-Ginzburg model reads:

$$\begin{aligned} V &= |dh|^2 + |\xi(\sigma)|^2 + |\,[A_1,\sigma]\,|^2 + |\,[\sigma,\overline{\sigma}]\,|^2 \\ &= 2|dW|^2 + |\mu|^2 + |\xi(A_1)|^2 + |\xi(\sigma)|^2 + |\,[A_1,\sigma]\,|^2 + |\,[\sigma,\overline{\sigma}]\,|^2, \end{aligned}$$

where $|.|$ denotes the norm associated to $g_{i\bar{j}}$ on M. This follows straightforwardly from writing out the lengthy 1-dimensional Lagrangian in component fields, as can be found in [11].

Classically, a supersymmetric vacuum corresponds to a field configuration for which the scalar potential ξ vanishes. From the expression above, it is clear that such vacua correspond to semistable critical orbits of W. Suppose first that we have a free critical orbit, where the stabilizer P is trivial. Then the components ξ_a are linearly independent along that orbit and $|\xi(A_1)|^2 + |\xi(\sigma)|^2$ is nonzero, so all components of A_1 and σ are massive.

$|\mu|^2 = 0$ only on the subspace $\mathcal{O} \subset \mathcal{O}_{\mathbb{C}}$ by definition of $\mathcal{O}$, so this term gives mass to field configurations that are not in $\mathcal{O}_{\mathbb{C}} - \mathcal{O}$. Since W is nondegenerate by assumption, the $|dW|^2$ term gives masses to all the fluctuations normal to $\mathcal{O}_{\mathbb{C}}$. We conclude that a stable critical orbit contributes one classical vacuum and in expanding the theory around such a vacuum, we only have massive fluctuations. Hence we can again conclude from a more physical point of view that a stable critical orbit contributes just one state to the equivariant cohomology of N.

However, suppose now that P is non-trivial, that is, $\mathcal{O}$ is isomorphic to the subgroup G/P of positive dimension. Then we can pick a point $p \in \mathcal{O}$ by (partially) fixing a gauge, which leaves an unbroken gauge group P. Fluctuations away from $\mathcal{O}$ are still massive, since $|\mu|^2 \neq 0$ there. But now A_1 and σ have flat directions on $\mathcal{O}$, since there is still an unbroken subgroup of the original gauge group: there is still a nontrivial subspace $\mathcal{O}$ on which W is minimal. Such flat directions mean that A_1, σ are massless and cause infrared divergences, as their propagator has a pole at zero momentum.

5.4 Localization in the open gauged σ-model

Analogous to section 4.4, we now consider the 1d gauged open Landau-Ginzburg governing maps $\Phi : L \longrightarrow M_{\mathbb{C}}$, with $L = (-\infty, 0] = \mathbb{R}^-$. We emphasize that $M_{\mathbb{C}}$, as always, comes from a complexification. Our goal is to show how we can localize this model on a path integral with an exotic integration cycle for 0-dimensional gauged quantum mechanics on M. Schematically, we want to find $\Psi_{top} \in H^\bullet_{G,h<\infty}(M_{\mathbb{C}})$ such that

$$\int_{\Gamma_{\mathcal{O}}} \Omega \exp S = (\overline{\Psi}, \Psi_{top}) = \int_{M_{\mathbb{C}} \times \mathfrak{g} \times \mathfrak{g}_{\mathbb{C}}} \mathcal{D}\mathcal{X}\eta[\Gamma_{\mathcal{O}}] \wedge \Omega \exp S, \tag{5.4.1}$$

*To see why, recall the Morse function $h = A_1^a \mu_a + 2\text{Re}\, W$. Now $2\text{Re}\, W$ has Morse index equal to one half of the real codimension of $\mathcal{O}$: this follows from the holomorphicity of W and the argument presented in section 4.2; hence its Morse index equals $\frac{1}{2}\dim M - \dim \mathcal{O}$. Now the first term $A_1^a \mu_a$ is a function on $\mathcal{O}_{\mathbb{C}} \times \mathfrak{g}$ and has a critical set defined by $\mu = \xi(A_1) = 0$ which leaves only $\mathcal{O} \subset \mathcal{O}_{\mathbb{C}}$ unfixed: hence its Morse index is exactly $\dim \mathcal{O}$. Hence the Morse index of h equals $\frac{1}{2}\dim M$. Note that this is independent of the dimension of P.

where $\Gamma_{\mathcal{O}} \subset M_{\mathbb{C}}$ is an exotic integration cycle. The subscript $\mathfrak{g}_{\mathbb{C}}$ indicates the integration over $\sigma, \overline{\sigma}$.

We assume $M_{\mathbb{C}}$ to be Calabi-Yau. The boundary condition on Φ is that $\Phi(s = -\infty)$ should lie in a semistable critical orbit of the superpotential h, with a choice of gauge fixing. Recall that the flow equations (5.2.4) and (5.2.5) defined Morse flow on $M_{\mathbb{C}} \times \mathfrak{g}$. With this boundary condition, the path integral on L computes a state

$$\Psi_{top} = \int_{\phi(-\infty)\in\mathcal{O}} \mathcal{D}\mathcal{X} \exp \mathcal{S}_{LG} \tag{5.4.2}$$

in the equivariant cohomology of $K = M_{\mathbb{C}} \times \mathfrak{g}$. Here $\mathcal{X}$ represents the field content of the theory. Associated to $\mathcal{O}$ is a cycle $C_{\mathcal{O}} \subset M_{\mathbb{C}} \times \mathfrak{g}$ determined by downward flow, its codimension is the index of the Morse function $\frac{1}{2}\dim M_{\mathbb{C}}$ (which we showed above). Analogous to (4.4.4), Ψ_{top} calculates the Poincaré dual to $C_{\mathcal{O}}$, which we denote by $\eta[\mathcal{C}_{\mathcal{O}}]$. One immediate problem is that $\mathcal{C}_{\mathcal{O}}$ has the wrong dimension $\frac{1}{2}\dim M_{\mathbb{C}} + \dim \mathfrak{g}$: we need to kill another $\dim \mathfrak{g}$ dimensions in a consistent way. It turns out that this is achieved by simply setting $A_1 = 0$, which is compatible with the boundary conditions at $s = 0$ and kills $\dim \mathfrak{g}$ degrees of freedom.

Now we can repeat our approach from section (4.4.1) to represent an 0-dimensional path integral over a middle-dimensional cycle in $M_{\mathbb{C}}$ as a 1-dimensional path integral. We can now pair Ψ_{top} with the canonical holomorphic top-form Ω on $M_{\mathbb{C}}$ and the complexified action is a H-invariant holomorphic function S on $M_{\mathbb{C}}$. Then a path-integral representation of the ordinary integral (5.1.2) is given by the formal equivalence

$$\int_{\Gamma_{\mathcal{O}}} \Omega \exp S = \left(\overline{\Psi}, \Psi_{top}\right) = \int_{M_{\mathbb{C}}\times\mathfrak{g}\times\mathfrak{g}_{\mathbb{C}}} \mathcal{D}\mathcal{X} \Psi_{top} \wedge \overline{\Psi} = \int_{M_{\mathbb{C}}\times\mathfrak{g}\times\mathfrak{g}_{\mathbb{C}}} \mathcal{D}\mathcal{X} \eta[\mathcal{C}_{\mathcal{O}}] \wedge \overline{\Psi}, \tag{5.4.3}$$

where $\Gamma_{\mathcal{O}} = C_{\mathcal{O}} \cap \{A_1(0) = 0\}$ represents a middle-dimensional cycle in the complexified phase space $M_{\mathbb{C}}$ of 0-dimensional gauged quantum mechanics.* Here $\overline{\Psi}$ is a state in $\overline{H}^{\bullet}_{G,h<\infty,c}(M_{\mathbb{C}} \times \mathfrak{g})$, which contains states in the equivariant cohomology with compact support along A_1 and σ. This is done since the scalar potential potentially contains flat directions for A_1, σ. With this additional assumption, the pairing (5.4.3) is formally finite, moreover, $\Gamma_{\mathcal{O}}$ does have the right dimension $\frac{1}{2}\dim M_{\mathbb{C}}$. Explicitly, $\overline{\Psi}$ is determined by the boundary conditions on the fields at $s = 0$. By arguments similar to the ones at the end of section (4.4), half of the fermions of the chiral and vector multiplets have to vanish at $s = 0$, as do the bosons $A_1, \sigma, \overline{\sigma}$. This determines

$$\overline{\Psi} = \left[\exp S \cdot \delta(\psi^{(1,0)})\delta(\chi_{(0,1})\delta(\lambda_1)\delta(\eta)\delta(A_1)\delta(\sigma)\delta(\overline{\sigma})\right]|_{s=0}. \tag{5.4.4}$$

Here λ, η are half of the fermions from the vector multiplet. All details of the full 'proof' of (5.4.3) can be found in [12] as an extension of the argument of section 4.4.

*$C_{\mathcal{O}}$ is of codimension $\frac{1}{2}\dim M_{\mathbb{C}}$, so is of dimension $\frac{1}{2}\dim M_{\mathbb{C}} + \dim G$. Setting $A_1(0) = 0$ fixes $\dim G$ dimensions at $s = 0$. Hence the intersection $C_{\mathcal{O}} \cap \{A_1(0) = 0\}$ has the correct dimension $\frac{1}{2}\dim M_{\mathbb{C}}$. Note that the cycle $\Gamma_{\mathcal{O}}$ is a G-invariant cycle and can be understood as the flow line given by solving the flow equation for $h = 2\text{Re}\, W$. In the gauged σ-model, we had the Morse function $h = 2\text{Re}\, W + A_1^a(0)\mu_a$, which gives the same Morse function if $A_1(0) = 0$.

6

Exotic duality: quantum mechanics and the A-model

In the previous chapter, the use of Morse flows enabled us to find *exotic integration cycles* that allowed us to re-express path integrals. In this chapter we come to our first application, we dualize the path integral of quantum mechanics

$$\int_{L\mathcal{M}} \mathcal{D}p(t)\mathcal{D}q(t) \exp\left(i\int (pdq - H(p,q)dt)\right) \prod_i O_i(t_i), \tag{6.0.1}$$

where the $O_i(t_i) = \exp(iHt_i)\, O_i \exp(-iHt_i)$ are observables of the quantum mechanical model and $(\mathcal{M},\omega)$ is the classical phase space. Here, we need to find an exotic integration cycle in the infinite-dimensional free loop space $L\mathcal{M}_{\mathbb{C}}$ over complexified phase space.

For trivial Hamiltonian, we will find that the open A-model path integral with suitable operator insertion computes exactly this quantum mechanical path integral with an exotic integration cycle. We will discuss what modifications are needed for non-trivial Hamiltonians and illustrate this duality with some concrete examples, in most detail for the quantum mechanical harmonic oscillator.

6.1 Time-independent quantum mechanics

We shall first consider the case with trivial Hamiltonian $H = 0$ on a phase space $\mathcal{M}$ of dimension $2n$. Then the system is time-independent and the general path integral is of the form

$$\operatorname{tr}_{\mathcal{H}} O_1 O_2 \dots O_N = \int_{L\mathcal{M}} \prod_{i=1}^{n} \mathcal{D}p_i(t)\mathcal{D}q_i(t) \exp\left(i\oint p_i dq_i\right) O_1(t_1)\dots O_N(t_N), \tag{6.1.1}$$

where the $p_i(t), q_i(t)$ are now periodic functions of $t \in S^1$ that are local coordinates of a map $\Phi : S^1 \to L\mathcal{M}$ in the free loop space $L\mathcal{M} = C^\infty(S^1,\mathcal{M})$, the space of smooth maps of circles into $\mathcal{M}$. $O_i(t_i) \equiv O_i(\Phi(t_i))$ are *functions* that are associated to the *operators* $O_i(t_i)$. Since there is no time evolution the only thing that matters is the cyclic ordering of the functions O_i inside the path integral.

Note that if we do not insert operators, the path integral will calculate the partition function $Z = \operatorname{tr}_{\mathcal{H}} 1 = \dim \mathcal{H}$, which just computes the dimension of the Hilbert space of physical states $\mathcal{H}$ associated to quantization of $\mathcal{M}$. It is well-known that $\dim \mathcal{H} < \infty$ iff $\mathcal{M}$ is compact. So to get a useful answer, the system can have only a finite number of degrees of freedom, which generically only happens for topological theories, which generically have $H = 0$. We consider this case first.

Complexification

Combining the p^i and q^i into a new variable x^j. Since $\omega = \sum_i dp^i \wedge dq^i$ is closed, we can locally write it as the curvature of an abelian 1-form gauge field b with curvature $\omega = db$. The path integral (6.1.1) then becomes

$$\operatorname{tr}_{\mathcal{H}} O_1 O_2 \dots O_N = \int_{L\mathcal{M}} \prod_{j=1}^{2n} \mathcal{D}x_j(t) \exp\left(i\oint b_j dx^j\right) O_1(t_1)\dots O_N(t_N).^{*} \tag{6.1.2}$$

*Note that for this expression to be well-defined, we need that the integrand is single-valued, most importantly $\exp\left(i\oint b_j dx^j\right)$ should be, so we would have $\oint b_j dx^j = 2\pi k, k \in \mathbb{Z}$. Normally, this would amount to a Dirac-quantization-like requirement on the connection with gauge field b. However, we are interested in finding an exotic integration cycle, which will also make sense when no such Dirac-quantization holds.

Just in the previous chapter, we complexify $\mathcal{M}$ to $\mathcal{M}_{\mathbb{C}}$, which is a $2n$ *real-dimensional complex manifold*, equipped with an integrable complex structure J and a notion of complex conjugation, an antiholomorphic involution τ.* We will assume that the fixed point set of τ is exactly $\mathcal{M}$. Furthermore we define a symplectic structure on $\mathcal{M}_{\mathbb{C}}$ whose real part coincides with ω on $\mathcal{M} \subset \mathcal{M}_{\mathbb{C}}$:

$$\Omega = \omega + ia, \tag{6.1.3}$$

where the closedness and non-degeneracy of Ω imply that a, ω are separately closed and non-degenerate. Here by ω we mean a form that coincides with the original ω on $\mathcal{M}$, for notational convenience. As an assumption, $\tau(\Omega) = -\overline{\Omega}$, so that $\tau^*(a) = a, \tau^*(\omega) = -\omega$, so for consistency we need $\omega|_{\mathcal{M}} = 0$. On $\mathcal{M}_{\mathbb{C}}$ we have local complex coordinates $(X^A, \overline{X}^A), A = 1, \dots, n$ on $\mathcal{M}_{\mathbb{C}}$ and a set of local real coordinates ξ^A (for instance, we can take $\xi^{2k-1} = \text{Re } X^k$ and $\xi^{2k} = \text{Im } X^k$). The map Φ is then described in local coordinates by $\xi^A(t)$.

Ω is the curvature of a complex-valued gauge field $\Lambda = \sum_{A=1}^n \Lambda_A d\xi^A$, which has real and imaginary parts

$$\Lambda = b - ic, \qquad a = -dc, \qquad \omega = db,$$

where b again should be understood as an extension of the original b. Here we can impose the additional condition that $c|_{\mathcal{M}} = 0$ so that the real part of the connection on $\mathcal{M}$ is flat, $dc|_{\mathcal{M}} = a|_{\mathcal{M}} = 0$, and we assume that only the (equivalence class) of the trivial connection satisfies this constraint.

Finally, we need analytic continuations of the function O_j that do not grow too fast (exponentially) at complex infinity in order for the path integral to converge. We shall assume that such an analytical continuation exists; generically we can think of the O_j being polynomials in the coordinates $\xi^A(t)$. Now that we have analytically continued all relevant objects, we want to express the original path integral (6.1.1) over the loop space $\mathcal{U}$ of classical phase space as a path integral over a middle-dimensional subspace of the total loop space $L\mathcal{M}_{\mathbb{C}}$:

$$\int_{\mathcal{C}_V \subset L\mathcal{M}_{\mathbb{C}}} \mathcal{D}\xi^A(t) \exp\left(i \oint \Lambda_A d\xi^A\right) O_1(t_1) \dots O_N(t_N) \tag{6.1.4}$$

on which this path integral formally converges.

Finding the exotic integration cycle

Referring back to our example with the Airy function, we want to take the real part of the action

$$h \equiv \text{Re } i \oint \Lambda_A d\xi^A = \oint_{\partial\Sigma} c_A d\xi^A \tag{6.1.5}$$

as our Morse function. Here Σ will be the worldsheet of the dual σ-model, which will satisfy $\partial\Sigma = S^1$. h is unbounded from above and below: if we redefine ξ^A using some $\lambda \in \mathbb{R}$ as

$$\xi^A(t) \longmapsto \xi'^A(t) = \xi^A(\lambda t) \Longrightarrow d\xi^A(t) \longmapsto d\xi'^A(t) = \lambda d\xi^A(t) \tag{6.1.6}$$

we see that we can arbitrarily rescale h. We recall that h should decay to $-\infty$ at complex infinity to keep the path integral formally convergent, in particular, h should be bounded from above on our integration cycle. Our first step is to find the critical points of $h = \oint c_A d\xi^A$, which follows from solving $\delta h = 0$.

$$\delta h = \oint_{\partial\Sigma} \delta\xi^A a_{AB} d\xi^B = \oint_{\partial\Sigma} \delta\xi^A a_{AB} \frac{d\xi^B}{dt} dt = 0, \tag{6.1.7}$$

we recall that $a = dc$. If a is non-degenerate and non-trivial, $\delta h = 0$ will vanish for any $\delta\xi^A$ only if $d\xi^B(t) = 0$ for all t. Hence, critical points of h correspond to *constant maps* $\Phi : S^1 \longmapsto \{p\} \in \mathcal{M}_{\mathbb{C}}$, which agrees with intuition since the Hamiltonian is trivial (Hamilton's equations of motion then say that all coordinates are independent of time). This implies that the space of critical points is a copy $\mathcal{M}^*_{\mathbb{C}}$

*On flat $\mathbb{C}^n$, τ corresponds to ordinary complex conjugation, which indeed is an involution: $\tau^2 = 1$.

of $\mathcal{M}_\mathbb{C}$ embedded in the loop space $L\mathcal{M}_\mathbb{C}$ of $\mathcal{M}_\mathbb{C}$. So to get an exotic integration cycle, we consider flow lines that start at a middle dimensional cycle $V \subset \mathcal{M}^*_\mathbb{C}$. To find these flow lines we need to solve the flow equations for $\xi^A(t)$. Introducing a flow parameter $s \in \mathbb{R}_- = (-\infty, 0]$, now the objects that are 'flowing' are functions $\xi^A(s,t)$ that are maps from the half-cylinder C to $\mathcal{M}_\mathbb{C}$, $\xi^A : C = \mathbb{R}_- \times S^1 \longrightarrow \mathcal{M}_\mathbb{C}$. This is a slight generalization of previous situations, where the 'flowing objects' were fixed position coordinates.

To formulate the flow equations for the maps ξ^A on the free loop space $L\mathcal{M}_\mathbb{C}$, we need a metric on its tangent space $TL\mathcal{M}_\mathbb{C}$, recall for instance the generic form in (4.2.7). For any loop $\xi \in L\mathcal{M}_\mathbb{C}$, a tangent vector is a section in $\Gamma(\xi^* T\mathcal{M}_\mathbb{C})$. Picking a metric g_{AB} on $\mathcal{M}_\mathbb{C}$ induces a metric on $TL\mathcal{M}_\mathbb{C}$ by setting

$$ds^2 = \oint_{S^1} dt g_{AB}(\xi(t)) \delta\xi^A(t) \delta\xi^B(t). \tag{6.1.8}$$

Here, by $\delta\xi^A(t)$ we denoted a 1-form on $L\mathcal{M}_\mathbb{C}$. Note that (6.1.8) includes an integral over time, so that it correctly is a map $G : TL\mathcal{M}_\mathbb{C} \times TL\mathcal{M}_\mathbb{C} \to \mathbb{R}$. Now (6.1.8) converts a 1-form $\delta\xi^A(t)$ into its dual vector $g^{AB}\frac{\partial}{\partial\xi^B(t)}$ while killing the integral over t. The variation of a function $h[\xi^A(t)]$ is given by $\frac{\delta h}{\delta\xi^B(t)}\delta\xi^B(t)$, whose dual is $g^{AB}\frac{\delta h}{\delta\xi^B(t)}\frac{\delta}{\delta\xi^A(t)} = \text{grad}\, h$. Introducing a flow parameter s the flow equation is therefore written as

$$\frac{d\xi}{ds} = \frac{d\xi^A(s,t)}{ds}\frac{\delta}{\delta\xi^A(s,t)} = -\text{grad}\, h \implies \frac{\partial\xi^A(s,t)}{\partial s} = -g^{AB}\frac{\delta h}{\delta\xi^B(s,t)} = -g^{AB}a_{BC}\frac{\partial\xi^C(s,t)}{\partial t}.$$

The boundary condition at $s \to -\infty$ is that $\xi^A(s,t)$ should limit, independently of t, to a point in the middle dimensional subspace $V \subset \mathcal{M}^*_\mathbb{C}$, and that $\xi^A(-\infty, t)$ is regular. The downward flow lines determined by this flow equation then furnish an exotic integration cycle $\mathcal{C}_V$ for the path integral (6.1.4). Since there is only 1 critical subset of h here, we do not have to worry about interpolation issues between multiple critical points (subsets).† Now we choose the metric g_{AB} such that it gives a compatible triple (I, g, a) such that

$$I^A{}_C = g^{AB}a_{BC} \tag{6.1.9}$$

is an almost complex structure. The flow equations then simplify to

$$\frac{\partial\xi^A(s,t)}{\partial s} = -I^A{}_C\frac{\partial\xi^C(s,t)}{\partial t}. \tag{6.1.10}$$

If I is integrable these are the Cauchy-Riemann equations for $\xi^A(s,t)$, with holomorphic coordinates $w = s + it$.‡ If I is *not* integrable, the flow equations (6.1.10) are the defining equation for an I-pseudo-holomorphic map. For this class of ξ^A, although I is not necessarily integrable, the flow equation is well-behaved and elliptic. In particular the flow equations are invariant under conformal mappings, so we can set $z = \exp w = \exp(s + it)$, which maps $w \in \mathcal{C} \longmapsto z \in D^*$, where D^* is the unit disk in the complex plane minus the origin. The fact that $\xi^A(-\infty, t)$ was well-defined and independent of t means that $\xi^A = \xi^A(z)$, as a function of z, extends continuously over $z = 0$ to a map $D \longrightarrow \mathcal{M}_\mathbb{C}$.

After this conformal transformation the exotic integration cycle is defined by the boundary values at $s = 0$ of all I-pseudoholomorphic maps $\Phi : D \longrightarrow \mathcal{M}_\mathbb{C}$ where $\Phi(z = 0)$ sits in the middle-dimensional cycle $V \subset \mathcal{M}^*_\mathbb{C}$.

†Moreover, the flow line is stable under perturbations of the metric g_{AB}: if we perturb our choice of metric, the resulting cycle $\mathcal{C}'_V$ determined by downward flow will be homologically equivalent to $\mathcal{C}_V$, since we cannot change global properties of homology cycles by small perturbations.

‡Locally we can always choose a basis of $\mathcal{M}_\mathbb{C}$ such that I is represented by a $\dim_\mathbb{R}\mathcal{M}_\mathbb{C} \times \dim_\mathbb{R}\mathcal{M}_\mathbb{C}$ matrix with 2×2-blocks on the diagonal of the standard complex structure on $\mathbb{R}^2$. In every 2×2-subspace the flow equations then look like the standard Cauchy-Riemann equations

$$\frac{\partial\xi^i}{\partial s} = \frac{\partial\xi^{i+1}}{\partial t}, \qquad \frac{\partial\xi^{i+1}}{\partial s} = -\frac{\partial\xi^i}{\partial t}, \qquad \frac{\partial Z^i}{\partial w} = 0, \tag{6.1.11}$$

which tells us $Z^i = \xi^i + i\xi^{i+1}$ is a holomorphic function of $w = s + it$.

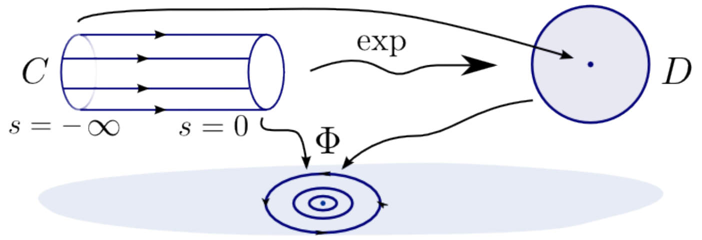


Figure 7: Embeddings of the semi-infinite flow cylinder C and the disk D into $\mathcal{M}_{\mathbb{C}}$, where $\partial C = \partial D = S^1$ represents a loop over which we integrate in (6.1.4).

Note that we are applying Morse theory on the infinite-dimensional space of loops on $\mathcal{M}_{\mathbb{C}}$. What is the validation of this? It is the elliptic nature of the flow equations: the Cauchy-Riemann equations are elliptic.

We now want to analyze the structure of $\mathcal{M}_{\mathbb{C}}$ further. Note that Ω is a holomorphic 2-form, it is a form of type $(2,0)$, with respect to the integrable complex structure J on $\mathcal{M}_{\mathbb{C}}$, so $\omega = \text{Im}\,\Omega$ is of type $(2,0)\oplus(0,2)$. $I = g^{-1}\omega$ however, cannot be of type $(2,0)$, as the metric g cannot be of type $(2,0)$ in J. Hence, we see that I and J have to be inequivalent complex structures. One natural way in which such a situation is feasible is when $\mathcal{M}_{\mathbb{C}}$ is an *almost hyperkähler manifold*: in that case we can choose I, J as part of a triple of almost structures I, J, K on M, of which at least J is integrable and we set $IJ = K$. Especially I, K do not have to be integrable. Also note that dimensions always work out: the complexified phase space $M_{\mathbb{C}}$ always has $2\cdot 2n = 4n$ dimensions, as required for (almost) hyperkähler manifolds.

To summarize, if we pick I such that ω_I defined through the compatibility equation $I = g^{-1}\omega_I$ is of type $(1,1)$, we see that the integration cycle $\mathcal{C}_V$ is a subset of all possible boundary values of I-pseudo-holomorphic disk embeddings into $\mathcal{M}_{\mathbb{C}}$. The A-model can be localized exactly on I-pseudoholomorphic maps. Combining all this, we deduce that the quantum mechanical path integral (6.0.1) can be expressed as an A-model path integral with a suitable operator insertion to impose the constraint that $\Phi(0)\in V$. Note that this operator insertion will be enforced on us by the selection rules for the A-model.

The dual A-model path integral

With this information, we have shown that a certain open A-model path integral is dual to (6.0.1). Namely, the localization property and the choice of worldsheet allows us to uniquely identify the dual A-model path integral as

$$\underbrace{\int_{\mathcal{C}_V\subset L\mathcal{M}_{\mathbb{C}}} \mathcal{D}\xi^A(t)\left(\exp\left(i\oint \Lambda_A d\xi^A\right)\prod_i \mathcal{O}_i\right)}_{\text{QM path integral over exotic cycle}}$$
$$= \underbrace{\int \mathcal{D}\xi\mathcal{D}\chi\mathcal{D}\psi \exp\left(-\frac{1}{\epsilon}I_A^{\text{top}}\right)\cdot\mathcal{O}_V(z=0)\left(\exp\left(i\oint\Lambda_A d\xi^A\right)\prod_i\mathcal{O}_i\right)}_{\text{A-model path integral}}, \qquad (6.1.12)$$

where

$$-I_A^{top} = -\int_D \Upsilon_A\wedge *\Upsilon^A - i\int_D \chi_A\wedge\mathcal{D}\psi^A + \frac{1}{4}\int_D R_{ABCD}\psi^A\psi^B\chi^C\wedge\chi^D \qquad (6.1.13)$$

is the Q-exact A-model action shifted by a constant term and we wrote the flow equations (6.1.10) in form notation as $\Upsilon^A = d\xi^A - *I^A_B d\xi^B = 0$, where $*$ is the Hodge star acting as $*ds = dt, *dt = -ds$. It is clear that this path integral localizes on $\Upsilon^A = 0$ as $\epsilon \longrightarrow 0$, which implies that only solutions to the flow equation contribute to the path integral upon localization. Here the superscript 'top' refers to the fact that we included the first half of (C.2.13), after performing the Bogomolny trick. Again, it is also obvious from the fermionic supersymmetry variations of the A-model (3.3.3) that this model localizes on the flow equations $Y^A = 0$.*

The operator insertion $\mathcal{O}_V(z=0)$ is the Poincaré dual $\eta(V)$ to V, which after localization leaves only those disk embeddings that map $z=0$ into V. Note that the degree of $\mathcal{O}_V$ is already determined by the index calculation (3.3.9), which shows that for a flat target space the degree of the operator insertion should $\dim_{\mathbb{C}} \mathcal{M}_{\mathbb{C}}$, that is, it is middle-dimensional in $\mathcal{M}_{\mathbb{C}}$.

After localization we are left with the integration over the boundary of all holomorphic disks that contribute as $\epsilon \longrightarrow 0$: those disks that have their center mapped into V. The residual path integral is just the quantum mechanical path integral with an exotic integration cycle.

Quantization and the A-model

At the special value $\epsilon = 1$ a straightforward calculation shows that the cross-terms in $Y_A \wedge *Y^A$ cancels against Re $i \oint \Lambda_A d\xi^A$. The idea is that in evaluating $Y^A \wedge *Y_A$, the cross term $2\omega_{AB}\frac{\partial \xi^A}{\partial s}\frac{\partial \xi^B}{\partial t}$ coming from the Bogomolny trick is exactly the Morse function $h = \text{Re}\,\Lambda$. This is explicitly shown in appendix B. This term then cancels with the factor of h contained in the boundary contribution $i \oint_{\partial C} \Lambda_A d\xi^A$. In that case we are left with the term $i \oint b_A d\xi^A$, which indicates that the open A-model is coupled to a brane: the boundary of the disk couples to a gauge field b that has curvature $\omega = db$. Since the support of this brane must be $\mathcal{M}_{\mathbb{C}}$ as the critical subset of h was a copy of $\mathcal{M}_{\mathbb{C}}$, this brane must be the canonical coisotropic brane $\mathcal{B}_{cc}$, as discussed in section 3.4.

Now suppose that we took as our worldsheet the finite cylinder $C_f = [-\tilde{s}, 0] \times S^1$, that is, we would only consider flow lines for some finite time. Then we can choose the boundary at $-\tilde{s}$ to lie on a Lagrangian rank-1 A-brane $\mathcal{B}_{\mathcal{L}}$, which must be supported on a submanifold of $\mathcal{M}_{\mathbb{C}}$, which is Lagrangian with respect to Im $\Omega = \omega$. Let us choose $\mathcal{L} = \mathcal{M}$† and $\mathcal{B}_{cc}$ in a way compatible with the symplectic form Im Ω on $\mathcal{M}_{\mathbb{C}}$. We shall describe the latter statement below. Using the categorical interpretation that topological A-branes are objects in the Fukaya category $\mathcal{F}^0(M)$ and open genus 0 A-model strings‡ are morphisms, as discussed in section A.4, this leads to the interpretation that at $\epsilon = 1$ we can think of the exotic A-model path integral as calculating a trace in the space Hom$(\mathcal{B}_{\mathcal{L}}, \mathcal{B}_{cc})$ of all open strings of genus 0 stretched between $\mathcal{B}_{\mathcal{L}}$ and $\mathcal{B}_{cc}$. As explained in [17] and [14], Hom$(\mathcal{B}_{\mathcal{L}}, \mathcal{B}_{cc})$ is exactly the Hilbert space $\mathcal{H}$ of physical states associated to the original phase space $\mathcal{M}$ in the A-model picture of quantization. With these prescriptions, the dual A-model path integral computes exactly the partition function of quantum mechanics on $\mathcal{M}$.

In the limit that $\tilde{s} \to \infty$, we recover the case that Σ is a disk D. The slight subtlety is that in this case the end of the cylinder at $\tilde{s} = \infty$ (the center of the disk) can lie in any middle-dimensional cycle V in $\mathcal{M}_{\mathbb{C}}$.

*A heuristic way to see that the A-model path integral is the correct one is to write a formula like:

$$\begin{aligned}&\int_{\mathcal{C}_V \subset L\mathcal{M}_{\mathbb{C}}} \mathcal{D}\xi^A(t) \exp\left(i \oint \Lambda_A d\xi^A\right) \prod_i O_i(t_i) \\ &= \int \mathcal{D}\xi^A(s,t) \delta\left(\partial_t \xi^A + I^A_B \partial_s \xi^B\right) \mathcal{O}\left(\xi^A(-\infty, t) \in V\right) \exp\left(i \oint \Lambda_A d\xi^A\right) \prod_i O_i(t_i) \end{aligned} \tag{6.1.14}$$

We promote the $\xi^A(t)$ to a function of two variables and put in a delta-functional that picks out only those maps Φ that satisfy the flow equation. The role of $\mathcal{O}$ should be clear. The first delta-functional can be written in terms of fields by using a Lagrange multiplier field T for the constraint $U^A = 0$. The path integral then becomes

$$\int \mathcal{D}T_A(s,t) \int_{\mathcal{C}_V} \mathcal{D}\xi^B(s,t) \exp\left(i \int_D T_A \wedge U^A\right) \ldots \tag{6.1.15}$$

Integrating out the T field by completing the square and adding fermions to cancel a determinant coming from the non-trivial argument in the delta-functional, one obtains exactly the A-model action (6.1.12), see also [12].

†Note that we chose $(\mathcal{M}_{\mathbb{C}}, \Omega)$ by construction such that $\mathcal{M}$ was Lagrangian in $\mathcal{M}_{\mathbb{C}}$ with respect to Im $\Omega = \omega$.

‡Note that here we are a bit ambiguous for brevity: by string, here we mean the A-model that is not coupled to worldsheet gravity, not the full topological string.

For any V the path integral can then be interpreted as some trace, however, it is in general not necessarily the trace in the space of physical states associated to quantization on $\mathcal{M}$. In our constructive approach however, in the limit $\tilde{s} \to \infty$ the dual A-model path integral clearly retains this property. Especially, we can freely choose V to exactly be the support of $\mathcal{B}_{\mathcal{L}}$. In that case, in the limit we retain the interpretation of the dual A-model path integral as the partition function of quantum mechanics on $\mathcal{M}$.

In the A-model picture of quantization, the space of physical states $\text{Hom}(\mathcal{B}_{\mathcal{L}}, \mathcal{B}_{cc})$ arises as follows. The idea is to start with a choice of complexification $(\mathcal{M}_{\mathbb{C}}, \Omega)$ of $(\mathcal{M}, \omega)$ such that $\text{Im}\,\Omega|_{\mathcal{M}} = \omega$. One then chooses $\mathcal{B}_{cc}$ such that it is an A-brane with respect to $\text{Im}\,\Omega$ as in (3.4.14). When the target space has a hyperkähler symmetry, this means generically that the curvature of the gauge field on $\mathcal{B}_{cc}$ must be $\text{Re}\,\Omega$. Complexification depends on a choice of complex structure I on $\mathcal{M}_{\mathbb{C}}$: this indicates that there can be a lot of inequivalent choices for $\mathcal{B}_{cc}$ on $\mathcal{M}_{\mathbb{C}}$. So generically, we want $\mathcal{B}_{cc}$ to be an A-brane with respect to the chosen complex structure I.

Regarding $\mathcal{M}_{\mathbb{C}}$ as a symplectic manifold with symplectic structure $\text{Im}\,\Omega$, we can pick a Lagrangian submanifold $\mathcal{L} = \mathcal{M}$. Quantizing the space $(\mathcal{B}_{\mathcal{L}}, \mathcal{B}_{cc})$ of open strings that end on $\mathcal{B}_{\mathcal{L}}, \mathcal{B}_{cc}$ gives $\mathcal{H} = \text{Hom}(\mathcal{B}_{\mathcal{L}}, \mathcal{B}_{cc})$.† An inner product on $\mathcal{H}$ can provided by using complex conjugation and CPT symmetry of the A-model. Note that the inherent ambiguity of this quantization procedure lies in the choice of $\mathcal{L}$. However, the most straightforward connection to familiar types of quantization comes from choosing $\mathcal{L} = \mathcal{M}$.

Analogously, the space of observables is furnished by $\text{Hom}(\mathcal{B}_{cc}, \mathcal{B}_{cc})$, which corresponds to the quantization of the space of holomorphic function $\mathcal{M}_{\mathbb{C}}$. The details of this entire procedure can be found in [14]. There is yet no full proof that quantization using the A-model is in general equivalent to other ways of quantization, such as geometric quantization or deformation quantization. However, it has been shown in [14] that the novel method reproduces known facts about $SU(2)$ and $SL(2,\mathbb{R})$ representation theory, by studying the A-model on the complexified 2-sphere.

To summarize: constructing the A-model path integral dual to the quantum mechanics on $\mathcal{M}$ actually gives a path integral derivation of the new way to define quantization of $\mathcal{M}$ by using the open A-model.

6.2 Including time-dependency

Now we treat the more general case where the Hamiltonian is non-trivial, so there is non-trivial time-dependent behavior of the system. As before, we want to compute traces, so we take the time parameter t to be periodic with period τ, so the expectation value of a set of observables becomes

$$\int_{L\mathcal{M}} \mathcal{D}p_i(t)\mathcal{D}q_i(t) \exp\left(i\int (p_i dq_i - H(p_i,q_i)dt)\right) O_1(t_1)O_2(t_2)\dots O_N(t_N). \tag{6.2.1}$$

We assume that the Hamiltonian can be analytically continued (note that any polynomial real function can be analytically continued and most Hamiltonians used are of that type); we write for the complexified Hamiltonian $\mathcal{H} = H_1 + iH_2$, where H_1 and H_2 are real. We emphasize that $\mathcal{H}$ is a complex function with respect to the complex structure J on $\mathcal{M}_{\mathbb{C}}$. After analytic continuation, we obtain

$$\int_{\mathcal{C}_V \subset L\mathcal{M}_{\mathbb{C}}} \mathcal{D}\xi^A(t) \exp\left(i\oint\left(\Lambda_A d\xi^A(t) - \mathcal{H}dt\right)\right) O_1(t_1)O_2(t_2)\dots O_N(t_N). \tag{6.2.2}$$

The Morse function is given by the real part of the action

$$h = \oint\left(c_A d\xi^A + H_2 dt\right). \tag{6.2.3}$$

We want to integrate over a middle-dimensional subspace $\mathcal{C}_V \subset L\mathcal{M}_{\mathbb{C}}$, determined by downward flow from a subspace V in the critical set of h. Before complexification, critical points of the action are given

†Specifically, one quantizes the zero modes of open strings that satisfy the right boundary conditions at their end points. This zero mode quantization corresponds to quantization of $\mathcal{M}$, with a prequantum line bundle $\mathcal{E}$.

by solutions to the classical equations of motion: solutions to the Hamilton equations. After complexification, critical points correspond to solutions to the real part of the complexified Hamilton equations; namely using

$$\delta h = \oint \frac{\delta c_A}{\delta \xi^B(s,t)} d\xi^A(s,t)\delta\xi^B(s,t) + \oint \frac{\delta H_2}{\delta \xi^B(s,t)} dt \delta \xi^B(s,t) \tag{6.2.4}$$

and $a = dc$, we find that critical points $\delta h = 0$ satisfy

$$a_{BA}\frac{d\xi^A(s,t)}{dt} = -\frac{\delta H_2}{\delta \xi^B(s,t)}, \tag{6.2.5}$$

which is the imaginary part of the complexified Hamilton equations

$$\Omega_{AB}\frac{d\xi^B(s,t)}{dt} = -\frac{\partial \mathcal{H}}{\partial \xi^A(s,t)}. \tag{6.2.6}$$

The flow equations become

$$\frac{\partial \xi^A}{\partial s} = -g^{AB}\frac{\delta h}{\delta \xi^B} = -g^{AB}\left(a_{BC}\frac{d\xi^C}{dt} + \frac{\partial H_2}{\partial \xi^B}\right) = -I^A{}_C\frac{d\xi^C}{dt} - \frac{\partial H_2}{\partial \xi_A}. \tag{6.2.7}$$

Here again we choose g_{AB} on $\mathcal{M}_\mathbb{C}$ such that we have the almost complex structure $I^A{}_C = g^{AB}a_{BC}$. But in this case the term involving H_2 perturbs the Cauchy-Riemann equation, therefore we need to extend the dual open A-model with a nontrivial superpotential W: this model is called the *open A-Landau-Ginzburg model*. In the A-Landau-Ginzburg model, the localization equations read:

$$\frac{\partial \phi^i}{\partial \overline{w}} + g^{i\bar{j}}\frac{\partial \overline{W}}{\partial \phi^{\bar{j}}} = 0. \tag{6.2.8}$$

where $(w, \overline{w})$ are worldsheet coordinates and $(\phi^i, \phi^{\bar{i}})$ are target space coordinates.* Using $w = s + it, \partial_{\overline{w}} = \frac{1}{2}(\partial_s + i\partial_t)$, the above equation can be written as

$$\frac{\partial \phi^i}{\partial s} = -i\frac{\partial \phi^i}{\partial t} - 2g^{i\bar{i}}\frac{\partial}{\partial \phi^{\bar{i}}}(W + \overline{W}). \tag{6.2.9}$$

Combining $\phi^i(s,t)$ and $\phi^{\bar{i}}(s,t)$ into the real coordinate $\xi^A(s,t)$, and using $(W + \overline{W}) = 2\text{Re}\,_I W$ we get

$$\frac{\partial \xi^A}{\partial s} = -I^A{}_B\frac{\partial \xi^B}{\partial t} - g^{AB}\frac{\partial}{\partial \xi^B}\left(4\text{Re}_I W\right) \tag{6.2.10}$$

upon which we make by comparison with (6.2.7) the identification

$$\text{Im}_J \mathcal{H} = H_2 = 4\text{Re}_I W, \tag{6.2.11}$$

where by the subscripts I, J we emphasized that $\mathcal{H}$ is J-holomorphic, while W is I-holomorphic. So if constraint (6.2.11) is satisfied, quantum mechanics with non-trivial Hamiltonian is dual to the open A-Landau-Ginzburg model.

This last observation implies that the duality only works if $\mathcal{M}_\mathbb{C}$ possesses special structure so (6.2.11) is satisfied. Since we can always define the almost complex structure $K = IJ$, $\mathcal{M}_\mathbb{C}$ generically must be *almost hyperkähler*: whereas J is integrable by assumption: I, K can be non-integrable almost complex structures.

*This equation needs proper interpretation when the canonical bundle of Σ is not trivial: interpreting ϕ^i as a scalar, the first term is a $(0,1)$-form on Σ, whereas the second term is a worldsheet scalar. In a local trivialization, we can identify these, but we cannot do so globally. In general, we should think of ϕ^i as a section of a line bundle, upon which such an equation makes sense globally. However, in our application Σ is the cylinder (see the discussion after (6.1.11)), whose canonical bundle is trivial, so we do not worry about this here.

In [12] it is shown that if $\mathcal{M}_{\mathbb{C}}$ is honestly hyperkähler, the generic case in which the duality holds is when $\mathcal{M}_{\mathbb{C}}$ has a symmetry generated by a Killing vector field V that preserves the hyperkähler structure and H is the moment map in complex structure I for this symmetry. Its Lie-derivative $\mathcal{L}_V = d\iota_V + \iota_V d$ annihilates the symplectic forms $\omega_I, \omega_J, \omega_K$ compatible with the complex structures I, J, K. Because a symplectic form by definition is closed $d\omega = 0$, for this requirement we only need that the interior contractions $\iota_V\omega_I, \iota_V\omega_J, \iota_V\omega_K$ are closed. Then, locally we can determine the moment maps (see section A.2) for the symmetry generated by V:

$$d\mu_I = \iota_V\omega_I \qquad d\mu_J = \iota_V\omega_J \qquad d\mu_K = \iota_V\omega_K. \tag{6.2.12}$$

It is a result (see reference [23] in [12]) that the quantity $\nu_I = \mu_J + i\mu_K$ is I-holomorphic, likewise $\nu_J = \mu_K + i\mu_I$ is J-holomorphic. So setting $\mathcal{H} = H_1 + iH_2 = i\nu_J = -\mu_I + i\mu_K$ automatically shows that we satisfy (6.2.11): we have $H_2 = \mu_K = 4\text{Re}\, W$ with $W = -\frac{i\nu_I}{4}$.

The idea of the proof is that the hyperkähler structure of $\mathcal{M}_{\mathbb{C}}$ and (6.2.11) are highly constraining. From the fact that $H_1 + iH_2$ and $\omega_K + i\omega_I$ are J-holomorphic, one finds that $(1 + iJ^t)(d(H_1 + iH_2) = 0$ and $J^t(\omega_K + i\omega_I = i(\omega_K + i\omega_I)$. From (6.2.11) one sees that $dH_2 = I^t dS$, where S is some I-holomorphic function. Now one can always find a vector field $\mathcal{V}$ that generates a symmetry that preserves the hyperkähler symmetry, such that (H_1, H_2, S) forms a triple of moment maps for $\mathcal{V}$. Explicitly, $\mathcal{V} = \omega_K^{-1} dH_2$.

In general, there may be topological obstructions for $\mathcal{M}_{\mathbb{C}}$ to admit such an almost hyperkähler structure. This can be understood from the point of view of holonomy. $\mathcal{M}_{\mathbb{C}}$ is a complex manifold of real dimension $4n$ by construction, so it possesses $U(4n)$-holonomy. In order to admit an almost hyperkähler structure, the structure group of the tangent bundle must be reducible to $Sp(n)$. So there can be topological obstructions that are measured by characteristic classes to admit such a reduction. We will not go into this further here, our examples will be hyperkähler from the start, so we do not have to worry about these subtle issues here.

Now the most simple examples of hyperkähler symmetries are given by various $U(1)$-actions. For instance, the simplest example is choosing $M = S^2$ and its complexification: the rotations generated by $SO(3)$ restrict to S^2. Taking rotations along some axis e, the associated moment map is just the spin about e [12]. Another example is the Taub-NUT space $\mathcal{T}$ (9.2.2) admits a hyperkähler symmetry by rotation of the circle fibers. Another example is given by toric hyperkähler manifolds (torus fibrations), they can arise as complexifications and also have hyperkahler symmetries induced by rotations on the torus fibers.

6.3 Dualizing the simple harmonic oscillator

We have seen that only Hamiltonians that are moment maps for a symmetry that preserves the hyperkähler structure on $\mathcal{M}_{\mathbb{C}}$ can satisfy (6.2.11). On flat $\mathbb{R}^2$, the simplest such symmetries are given by translations and rotations. For the latter, the moment map is given by the Hamiltonian of the simple harmonic oscillator, which is just $H = \frac{1}{2}\left(p^2 + q^2\right)$. Its partition function is standard: the Hamiltonian has spectrum $E_i = (i + \frac{1}{2})\hbar, i \in \mathbb{Z}_{\geq 0}$, from which we have in Euclidean signature

$$Z = \sum_{i=0}^{\infty} \exp\left(-\beta\left(i + \frac{1}{2}\right)\hbar\right) = \frac{\exp\left(-\beta\hbar/2\right)}{1 - \exp\left(-\beta\hbar\right)} = \frac{1}{2\sinh\left(\beta\hbar/2\right)} \overset{\beta \longrightarrow iT}{\longrightarrow} \frac{1}{2i\sin\left(\hbar T/2\right)}.$$

Here we will construct in detail the dual open A-Landau-Ginzburg model. The reference for part of this material is [15].

The open A-Landau-Ginzburg model

Since inserting observables for the simple harmonic oscillator does not change anything essential, we focus on just the partition function Z of the theory. We want to express Z in terms of the open A-Landau-Ginzburg model in the presence of a canonical coisotropic brane $\mathcal{B}_{cc}$, which governs embeddings

Φ of the half-cylinder $\mathcal{C} = \mathbb{R}_+ \times S^1$ into the complexified phase space $\mathcal{M}_\mathbb{C} = (\mathbb{R}^2)_\mathbb{C} = \mathbb{C}^2$. Our goal is to make sense of the equality

$$\begin{aligned} Z &= \int_{\mathcal{C}_V \subset L\mathcal{M}_\mathbb{C}} \mathcal{D}\xi^A(t) \exp\left(i \int_{S^1} \left(\Lambda_A d\xi^A - \mathcal{H}dt\right)\right) \\ &= \int_{\mathcal{A}} \mathcal{D}\xi^A(s,t)\mathcal{D}\psi_+\mathcal{D}\psi_- \exp\left(-S^{top}_{A-LG}\right) \exp\left(i \oint_{\partial\mathcal{C}} \left(\Lambda_A d\xi^A - \mathcal{H}dt\right)\right). \end{aligned} \tag{6.3.1}$$

Here $\tilde{S}$ represents the topological twisted action expressed of the open A-Landau-Ginzburg model and $\mathcal{A}$ represents a boundary condition on the worldsheet embeddings that are admitted in the path integral, which we will discuss below. Note that no operator insertion is necessary since the Euler number of the half cylinder is 0, which together with the flatness of $\mathcal{M}_\mathbb{C}$ makes the selection rule (3.3.9) trivial. Any optional operator insertions cannot carry any net axial R-charge. Twisting the model is not straightforward as it cannot always be done. For the simple harmonic oscillator, however, an appropriate twist is possible, which we discuss below.

We choose local coordinates on the target space are $(\phi^i, \phi^{\bar{i}}), i = 1, \ldots, 2n$ which are I-holomorphic / antiholomorphic, and there is a superpotential W, a holomorphic function of ϕ^i. The standard Landau-Ginzburg action is an extension of the $\mathcal{N} = (2,2)$ σ-model action:

$$\begin{aligned} S_{LG} &= \int_{\mathcal{C}} d^2w \left(2g_{i\bar{j}}\bar{\partial}\phi^i\partial\phi^{\bar{j}} + 2g^{i\bar{j}}\partial_i W \partial_{\bar{j}}\overline{W} + \psi^i_+\psi^j_- D_i\partial_j W + \psi^{\bar{i}}_+\psi^{\bar{j}}_- D_{\bar{i}}\partial_{\bar{j}}\overline{W}\right) \\ &+ \int_{\mathcal{C}} d^2w \left(\frac{i}{2}g_{\mu\nu}\psi^\mu_+ D_{\overline{w}}\psi^\nu_+ + \frac{i}{2}g_{\mu\nu}\psi^\mu_- D_w\psi^\nu_- + R_{i\bar{j}k\bar{l}}\psi^i_+\psi^{\bar{j}}_+\psi^k_-\psi^{\bar{l}}_-\right) \end{aligned} \tag{6.3.2}$$

and the fermionic supersymmetry variations are likewise extensions:

$$\begin{aligned} \delta\psi^i_+ &= -\tilde{\alpha}_-\partial_w\phi^i - i\alpha_+\psi^k_-\Gamma^i_{kl}\psi^l_+ - i\alpha_+ g^{i\bar{j}}\partial_{\bar{j}}\overline{W}, & \delta\psi^{\bar{i}}_+ &= -\alpha_-\partial_w\phi^{\bar{i}} - i\tilde{\alpha}_+\psi^{\bar{k}}_-\Gamma^{\bar{i}}_{\bar{k}\bar{l}}\psi^{\bar{l}}_+ - i\tilde{\alpha}_+ g^{\bar{i}j}\partial_j W, \\ \delta\psi^i_- &= -\tilde{\alpha}_+\partial_{\overline{w}}\phi^i - i\alpha_-\psi^k_+\Gamma^i_{kl}\psi^l_- + i\alpha_- g^{i\bar{j}}\partial_{\bar{j}}\overline{W}, & \delta\psi^{\bar{i}}_- &= -\alpha_+\partial_{\overline{w}}\phi^{\bar{i}} - i\tilde{\alpha}_-\psi^{\bar{k}}_+\Gamma^{\bar{i}}_{\bar{k}\bar{l}}\psi^{\bar{l}}_- + i\tilde{\alpha}_- g^{\bar{i}j}\partial_j W. \end{aligned}$$

Note that by redefining the phase of W one can account for unwanted factors of i.

Hyperkähler structure

Before we discuss the exotic A-twist, we first fix a hyperkähler on $\mathcal{M}_\mathbb{C}$. We take coordinates (p,q) on the non-compact phase space $\mathbb{R}^2$, which double to

$$(p_1, p_2, q_1, q_2) = (\xi^1, \xi^2, \xi^3, \xi^4), \tag{6.3.3}$$

which we view as real coordinates on the flat complex space $\mathbb{C}^2 \cong \mathbb{H}$. Using the quaternionic structure, $\mathbb{C}^2$ naturally has a hyperkahler structure generated by three complex structures I, J, K. In matrix representation I, J, K are given by

$$I = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{6.3.4}$$

for which holomorphic coordinates are

I	$u_1 = p_1 + iq_1 = \xi^1 + i\xi^3$	$\overline{u}_1$	$u_2 = p_2 - iq_2 = \xi^2 - i\xi^4$	$\overline{u}_2$
J	$z_1 = p_1 + ip_2 = \xi^1 + i\xi^2$	$\overline{z}_1$	$z_2 = q_1 + iq_2 = \xi^3 + i\xi^4$	$\overline{z}_2$
K	$v_1 = p_1 + iq_2 = \xi^1 + i\xi^4$	$\overline{v}_1$	$v_2 = p_2 + iq_1 = \xi^2 + i\xi^3$	$\overline{v}_2$

We follow here the conventions stated in the previous sections. Therefore, the simple harmonic oscillator Hamiltonian $H = \frac{1}{2}\left(p^2 + q^2\right)$ is complexified using J-holomorphic coordinates to $\mathcal{H} = \frac{1}{2}\left(z_1^2 + z_2^2\right)$. Its real part restricted to $p_2 = q_2 = 0$ obviously equals H. To satisfy (6.2.11), we have

$$\text{Im}\,\mathcal{H} = H_2 = 4\text{Re}\,W \Rightarrow H_2 = p_1p_2 - q_1q_2 \Rightarrow W = \frac{1}{4}u_1u_2. \tag{6.3.5}$$

We write the components of I, J, K-complex symplectic structures as

$$\Omega_I = \omega_J + i\omega_K, \qquad \Omega_J = \omega_I - i\omega_K, \qquad \Omega_K = \omega_I + i\omega_J.$$

$\Omega = \omega + ia = db - idc$ is J-holomorphic, so we can identify

$$\text{Re}\,\Omega = db = \omega_I = gI = I, \qquad \text{Im}\,\Omega = -dc = -\omega_K = -gK = -K. \tag{6.3.6}$$

To correctly identify b, c after the complexification in complex structure J, we need to write

$$\oint pdq = \frac{1}{2}\oint (pdq - qdp) \to \frac{1}{2}\oint (z_1 dz_2 - z_2 dz_1) = \oint (b - ic)_A d\xi^A.$$

Here we symmetrized by integration by parts, in order to correctly get all the components of ω_I, ω_K right. We then have that b, c can be identified as

$$b_A = \frac{1}{2}(-\xi^3, \xi^4, \xi^1, -\xi^2), \qquad c_A = \frac{1}{2}(\xi^4, \xi^3, -\xi^2, -\xi^1). \tag{6.3.7}$$

It is straightforward to check that $\mathcal{H}$ is indeed the moment map for rotations. The vector $\mathcal{V}$ that generates rotations is

$$\mathcal{V} = -z_1\frac{\partial}{\partial z_2} + z_2\frac{\partial}{\partial z_1} - \bar{z}_1\frac{\partial}{\partial \bar{z}_2} + \bar{z}_2\frac{\partial}{\partial \bar{z}_1}, \tag{6.3.8}$$

the flow of $\mathcal{V}$ is described by the system $\dot{z}_I\frac{\partial}{\partial z_I} = \mathcal{V}, I = 1, 2, \bar{1}, \bar{2}$. A representation in forms of Ω_J is $dz^1 \wedge dz^2$ (just the J-complexification of $dp \wedge dq$). We claim that $\mathcal{H} = i\nu_J, H_1 = -\mu_I, H_2 = \mu_K$, where $d\nu_J = \iota_\mathcal{V}\Omega_J$ and ν_J is J-holomorphic. For completeness

$$\Omega_I = du_1 \wedge du_2, \qquad \Omega_J = dz_1 \wedge dz_2, \qquad \Omega_K = dv_1 \wedge dv_2.$$

Using lower indices we have

$$\frac{1}{2}d\left(z_1^2 + z_2^2\right) = z_1 dz_1 + z_2 dz_2, \qquad \iota_\mathcal{V}\Omega_J = z_1 dz_1 + z_2 dz_2.$$

So writing $\mathcal{H} = i\nu_J$, we identify $\nu_J = -\frac{i}{2}\left(z_1^2 + z_2^2\right) = \mu_K + i\mu_I$, so $\mu_K = \text{Im}\,\mathcal{H} = H_2 = p_1 p_2 + q_1 q_2$. Note also that the matrix coefficients of $\omega_I = \text{Re}\,\Omega_J$ correctly correspond to I and $-\omega_K = \text{Im}\,\Omega_J$ likewise corresponds to $-K$. It is obvious that restricted to $\mathcal{M}$, Re $\mathcal{H}|_\mathcal{M} = H$ is the moment map for rotations generated by $\mathcal{V}|_\mathcal{M}$.

The exotic A-twist

The obstruction to a successful A-twist are the terms in (6.3.2) mixing fermions and the superpotential, including $\psi_+^{\bar{i}}\psi_-^{\bar{j}} D_{\bar{i}}\partial_{\bar{j}}\overline{W}$, which transforms after the standard A-twist (3.3.1) into $\psi_{\bar{z}}^{\bar{i}}\chi^{\bar{j}} D_{\bar{i}}\partial_{\bar{j}}\overline{W}$, which is no longer Lorentz-invariant on the worldsheet. There are basically two ways to remedy this problem: use the quasihomogeneity of W to introduce an additional twist, or multiply the $\psi\psi W$ term by another *holomorphic* section that makes the offensive term Lorentz invariant after twisting. Since our worldsheets will always be disks, the only holomorphic sections are constants, and so we cannot use this trick.

These methods are not equivalent, as not all situations allow the first twisting: it is still an open problem to find the generic situation in which an A-twist is possible. When the phase space $\mathcal{M}_\mathbb{C}$ can be seen as the total space of a fiber bundle, a modified A-twist is possible. We will demonstrate this for the simple harmonic oscillator.

Note that $\mathcal{M}_\mathbb{C} = \mathbb{C}^2$ admits a $U(1) \times U(1)$ group of rotations of the coordinates $(u_1, u_2) \in \mathbb{C}^2$, under which $W \longrightarrow e^{i\alpha}W$. We want to use this extra symmetry, whose generator we denote by $\tilde{Q}$ to introduce an additional twist. This approach is discussed in [15]: we can view $\mathbb{C}^2$ as the total space of the trivial line bundle $\mathbb{C} \longrightarrow \mathbb{C}$, where the general form of the superpotential is $W = p_a s^a, a = 1$, where p_a are fiber coordinates and s^a is a section of the dual bundle $\mathbb{C}^* \longrightarrow \mathbb{C}$. Here, we can take

$$\phi^p = u_1 \text{ as the fiber coordinate and } \phi^2 = u_2, \text{ as the base coordinate.} \tag{6.3.9}$$

thought of as a section of the dual space $\mathbb{C}^*$, which is isomorphic to $\mathbb{C}$. From now on, we take $i = 1 = p, \bar{i} = \bar{1} = \bar{p}$ as the fiber indices and $i = 2, \bar{i} = \bar{2}$ as the base index.

First, one can reformulate the twisting procedure from chapter 3 a bit, by using

$$Q_R = \frac{1}{2}(F_A - F_V), \qquad Q_R = -\frac{1}{2}(F_A + F_V). \tag{6.3.10}$$

In the standard A-twist the new bundles are then determined by tensoring the old ones with $L' = K^{-\frac{1}{2}Q_R}\overline{K}^{+\frac{1}{2}Q_L}$, whereas we tensor with $L'' = K^{+\frac{1}{2}Q_R}\overline{K}^{+\frac{1}{2}Q_L}$ for the B-twist. This is easily checked:

Generators	F_V	F_A	Q_R	Q_L	L	L'	A-twist $L \otimes L'$	B-twist $L \otimes L''$
Q_-, ψ_-	-1	1	0	1	$K^{1/2}$	$K^0 \otimes \overline{K}^{1/2}$	$\mathbb{C}$	K
$\overline{Q}_+, \overline{\psi}_+$	1	1	-1	0	$\overline{K}^{1/2}$	$K^{1/2} \otimes \overline{K}^0$	$\mathbb{C}$	$\mathbb{C}$
$\overline{Q}_-, \overline{\psi}_-$	1	-1	0	-1	$K^{1/2}$	$K^0 \otimes \overline{K}^{-1/2}$	K	$\mathbb{C}$
Q_+, ψ_+	-1	-1	1	0	$\overline{K}^{1/2}$	$K^{-1/2} \otimes \overline{K}^0$	$\overline{K}$	$\overline{K}$

Table 2: An overview of $U(1)$-charges and the new bundles after the A-twist and B-twist, using $K^{-1} \cong \overline{K}$.

This can be compared to table (2). The idea is now to use the extra $U(1)$ symmetry generated by $\tilde{Q}$ to do an extra twist, namely to define

$$Q'_R = Q_R - \tilde{Q}, \qquad Q'_L = Q_L - \tilde{Q} \tag{6.3.11}$$

and tensor the old bundles by $\tilde{L} = K^{-\frac{1}{2}Q'_R}\overline{K}^{\frac{1}{2}Q'_L}$. The choice here is to do the exotic A-twist only on the fiber coordinate, while the base coordinate gets the normal A-twist. All the $U(1)$-charges of the fields are listed below.

	Q_R	Q_L	$\tilde{Q}$	L	$Q'_R = Q_R - \tilde{Q}$	$Q'_L = Q_L - \tilde{Q}$	$L \otimes \tilde{L}$	New fields
ϕ^p	0	0	1	0	-1	-1	K	ϕ^p
$\phi^{\overline{p}}$	0	0	-1	0	1	1	$\overline{K}$	$\phi^{\overline{p}}$
ψ_+^p	1	0	1	$K^{\frac{1}{2}}$	0	-1	K	ψ_z^p
$\psi_+^{\overline{p}}$	-1	0	-1	$K^{\frac{1}{2}}$	0	1	$\mathbb{C}$	$\chi^{\overline{p}}$
ψ_-^p	0	1	1	$\overline{K}^{\frac{1}{2}}$	-1	0	$\mathbb{C}$	χ^p
$\psi_-^{\overline{p}}$	0	-1	-1	$\overline{K}^{\frac{1}{2}}$	1	0	$\overline{K}$	$\psi_{\overline{z}}^{\overline{p}}$
ϕ^2	0	0	0	0	0	0	$\mathbb{C}$	ϕ^1
$\phi^{\overline{2}}$	0	0	0	0	0	0	$\mathbb{C}$	$\phi^{\overline{1}}$
ψ_+^2	1	0	0	$K^{\frac{1}{2}}$	1	0	$\mathbb{C}$	χ^2
$\psi_+^{\overline{2}}$	-1	0	0	$K^{\frac{1}{2}}$	-1	0	K	$\psi_z^{\overline{2}}$
ψ_-^2	0	1	0	$\overline{K}^{\frac{1}{2}}$	0	1	$\overline{K}$	$\psi_{\overline{z}}^2$
$\psi_-^{\overline{2}}$	0	-1	0	$\overline{K}^{\frac{1}{2}}$	0	-1	$\mathbb{C}$	$\chi^{\overline{2}}$

Table 3: The exotic A-twist, regarding $\mathbb{C}^2$ as the total space of a fiber bundle, with the sleight of hand that $\overline{K}$ transforms like K^{-1}.

Note that the fiber coordinates now become Lorentz vectors and can no longer be used as observables anymore, as we would have to use worldsheet metrics to contract with the Lorentz indices. This seems a bit unnatural, we can only use the extra phase space coordinates as observables now (the old ones $u_1 = p_1 + iq_1$ are twisted to vectors here. This is a matter of choice though: we could also have chosen to view u_2 as the fiber coordinates, in which case the old phase space coordinates would remain good observables.

Now W is a holomorphic function of the ϕ^i, in particular, we have that $W = \frac{1}{4}\phi^p\phi^2 = \frac{1}{4}u_1u_2$. Under the new Lorentz group, the term $\ldots + \zeta_+^{\bar{i}}\zeta_-^{\bar{j}} D_{\bar{i}}\partial_{\bar{j}}\overline{W}$ sits in the bundle K^0, so is a Lorentz scalar on the worldsheet. This is easily checked: since the target space is flat, this term reduces to $\zeta_+^i\zeta_-^j\partial_i\partial_j W$. The only nonzero terms are

$$\frac{1}{4}\left(\psi_+^p\psi_-^2 + \psi_+^{\overline{p}}\psi_-^{\overline{2}} + \psi_+^2\psi_-^p + \psi_+^{\overline{2}}\psi_-^{\overline{p}}\right) = \frac{1}{4}\left(\psi_z^p\psi_{\overline{z}}^2 + \chi^{\overline{p}}\chi^{\overline{2}} + \chi^2\chi^p + \psi_z^{\overline{2}}\psi_{\overline{z}}^{\overline{p}}\right). \tag{6.3.12}$$

Using table (3), these are all worldsheet scalars again: note that the Lorentz index structure is consistent. It is an easy check that the fermion kinetic terms $\psi_+^{\overline{p}}D_z\psi_+^p + \ldots$ remain worldsheet scalars too. This makes the topological theory well-defined again. It is an easy check that all the other bosonic terms in the Lagrangian remain worldsheet scalars after the exotic twist.

The topological action and coupling to the $\mathcal{B}_{cc}$-brane

Expressing the action in coordinates $(s,t) \in (-\infty, 0] \times S^1$ with $w = s + it$ on the half-cylinder, one can check that at $\epsilon = 1$ cross-terms in the bosonic action cancel with part of the complex boundary couplings, which indicate the presence of the canonical coisotropic brane as in section 6.1.4. The appropriate topological twisted action of the open A-Landau-Ginzburg model is

$$\begin{aligned} S^{top}_{A-LG} = &\int_C d^2w \left(2g_{i\bar{j}}\left(\overline{\partial}\phi^i + g^{i\bar{k}}\partial_{\bar{k}}\overline{W}\right)\left(\partial\phi^{\bar{j}} + g^{\bar{j}k}\partial_k W\right) + \zeta_+^i\zeta_-^j D_i\partial_j W + \zeta_+^{\bar{i}}\zeta_-^{\bar{j}} D_{\bar{i}}\partial_{\bar{j}}\overline{W}\right) \\ &+ \int_C d^2w \left(\frac{i}{2}g_{\mu\nu}\zeta_+^\mu D_{\overline{w}}\zeta_+^\nu + \frac{i}{2}g_{\mu\nu}\zeta_-^\mu D_w\zeta_-^\nu + R_{i\bar{j}k\bar{l}}\zeta_+^i\zeta_+^{\bar{j}}\zeta_-^k\zeta_-^{\bar{l}}\right). \end{aligned} \tag{6.3.13}$$

where we denoted the twisted fermion fields collectively by ζ. The purely bosonic part contains cross-terms:

$$\begin{aligned} 2\int_C d^2w g_{i\bar{j}}\left(\overline{\partial}\phi^i + g^{i\bar{k}}\partial_{\bar{k}}\overline{W}\right)\left(\partial\phi^{\bar{j}} + g^{\bar{j}k}\partial_k W\right) = &2\int_C d^2w \left(g_{i\bar{j}}\overline{\partial}\phi^i\partial\phi^{\bar{j}} + g^{i\bar{j}}\partial_i W\partial_{\bar{j}}\overline{W}\right) \\ &- 2\int_C d^2w \left(\partial\overline{W} + \overline{\partial}W\right) \end{aligned} \tag{6.3.14}$$

where $\partial = \frac{1}{2}(\partial_s - i\partial_t), \overline{\partial} = \frac{1}{2}(\partial_s + i\partial_t)$. The cross-term simplifies to

$$2\int_C d^2w \left(\partial\overline{W} + \overline{\partial}W\right) = 2\int_{-\infty}^0 \int_{S^1} dsdt\, (2\partial_s \mathrm{Re}\, W - 2i\partial_t \mathrm{Im}\, W).$$

The second term is 0, by the periodicity of $t \in S^1$, whereas the first integral equals $4\oint_{\partial C} dt\mathrm{Re}\, W$ (we assume that Re W vanishes at $s = -\infty$). So we see that at $\epsilon = 1$ the term $4\oint_{\partial C} dt\mathrm{Re}\, W$ cancels against the boundary coupling $\oint_{\partial C} H_2 dt = 4\oint_{\partial C} \mathrm{Re}\, W$ in (6.3.1), using the identification (6.2.11).

Hence at $\epsilon = 1$ the path integral (6.3.1) simplifies to

$$Z = \int_{\mathcal{A}} \mathcal{D}\phi\mathcal{D}\zeta_+\mathcal{D}\zeta_- \exp\left(-S_{A,LG}\right)\exp\left(i\oint_{\partial C} b_\mu d\phi^\mu\right)\exp\left(-\int_{\partial C} iHdt\right).$$

It is clear that at this value of the coupling the model is again coupled to a canonical coisotropic brane through the gauge field b. It is a slightly surprising result that in the presence of a unitary coupling to $\mathcal{B}_{cc}$ there are no worldsheet instantons: in that case all the contributions come from perturbative calculations, as shown in [16, 17]. However, the result cannot depend on the value of ϵ, the result at $\epsilon = 1$ should correspond to the result obtained by localization $\epsilon \longrightarrow 0$. Note that at $\epsilon = 0$, worldsheet instantons are allowed again, as the model is not unitarily coupled (the coupling is $i\oint(b-ic)d\xi$) to $\mathcal{B}_{cc}$ anymore.

Because $\mathcal{M}_{\mathbb{C}}=\mathbb{C}^2$ has an additional hyperkähler structure, the classification of topological branes is richer, which we demonstrate here. The requirement for $\mathcal{B}_{cc}$ to be an A-brane in some complex structure X is that $\omega_X^{-1}db=\omega_X^{-1}\omega$ is an integrable complex structure. A B-brane is characterized by the fact that the curvature of its gauge field is of type $(1,1)$ with respect to X, see for instance [17]. The curvature of our brane is $db=\omega=\omega_I=gI=I$ since g is flat. One finds that

$$(-\omega_I^{-1}\omega_I)^2=+\text{id},\qquad(-\omega_J^{-1}\omega_I)^2=-\text{id},\qquad(-\omega_K^{-1}\omega_I)^2=-\text{id},$$

from which we see that $\mathcal{B}_{cc}$ is not an A-brane in complex structure I, but is in J,K. Note that for a vector $v^i\frac{\partial}{\partial x^i}$ to be of type $(1,0)$ with respect to a complex structure I means that $I_i^{\ j}v^i\frac{\partial}{\partial x^j}=v^j\sqrt{-1}\frac{\partial}{\partial x^j}$. However, for covectors w_jdx^j, one has to use the transpose of I: a covector or differential form is of type $(1,0)$ when $(I^t)^i_{\ j}w_idx^j=\sqrt{-1}w_idx^i$.

Hence, we check that in complex structure I, we have that $I^t\omega_I I=+\omega_I$ from which we see that $\omega=\omega_I$ is of type $(1,1)$ with respect to I, which indicates it is a B-brane in complex structure I. So our brane with $db=\omega_I$ is of type (B,A,A), in the terminology of [17]. This coincides perfectly with the general characterization of $\mathcal{B}_{cc}$ given there, with the proviso that relative to the notation there, I,J have been switched around here, so our (B,A,A)-brane corresponds to their (A,B,A)-brane. Note that integrability of I,J,K is trivial due to flatness of $\mathbb{C}^2$.

Note that this is compatible with the discussion in section 6.1.4. Our choice for $\mathcal{B}_{cc}$ meant choosing its curvature $db=\omega_I=\text{Re }\Omega$, which indeed gives an A-brane in the complex structure J and symplectic structure $\text{Im }\Omega$.

Finally, note that $\mathcal{L}=\{u_1=\xi^1+i\xi^3=0\}$ is Lagrangian with respect to $\text{Im }\Omega_J=\text{Im }(dz_1\wedge dz_2)=d\xi^1\wedge\xi^4+d\xi^2\wedge d\xi^3$. Hence it would be most convenient to use this in A-model quantization.

Localization

We have shown that there is a valid A-twist so we can consider localization properties of this model. Looking at the fermionic Q-fixed points

$$\delta\zeta_+^{\bar{i}}=-\alpha_-\partial\phi^{\bar{i}}-i\tilde{\alpha}_+g^{\bar{i}j}\partial_jW+\text{fermions},\qquad\delta\zeta_-^{i}=-\tilde{\alpha}_+\bar{\partial}\phi^i+i\alpha_-g^{i\bar{j}}\partial_{\bar{j}}\overline{W}+\text{fermions}.\qquad(6.3.15)$$

we see by setting $\tilde{\alpha}_+=\alpha_-$ that the model localizes on the flow equation of quantum mechanics with non-trivial Hamiltonian, namely $\bar{\partial}\phi^i-ig^{i\bar{j}}\partial_{\bar{j}}\overline{W}=0$. Note that the discrepancy between this formula and (6.2.8), as we mentioned before, can be lifted by redefining the phase of W.

Since there actually are two separate supersymmetry parameters $\tilde{\alpha}_+,\alpha_-$, one could argue that the model actually localizes to $\bar{\partial}\phi^i=dW=0$, as advertised in [15]. However, as we discussed in section 3.4, the boundary condition at $\partial\Sigma$ for A-type supersymmetry in the case that $\Sigma=\mathcal{C}$, becomes $\overline{G}_+^s+G_-^s=0$. Here the superscript s denotes the normal component of $G_\pm^\mu$ normal to $\partial\Sigma$. This boundary condition implies that only the combination of supersymmetry parameters $\alpha=\tilde{\alpha}_++\alpha_-$ is preserved at $\partial\Sigma$. Therefore, the model can only localize on the entire flow equation (6.3.15).

The flow equation reads explicitly

$$\bar{\partial}u_1-ig^{i\bar{j}}\partial_{\bar{j}}\frac{1}{4}\bar{u}_1\bar{u}_2=\bar{\partial}u_1-\frac{i}{2}\bar{u}_2=0,\qquad\bar{\partial}u_2-ig^{i\bar{j}}\partial_{\bar{j}}\frac{1}{4}\bar{u}_1\bar{u}_2=\bar{\partial}u_2-\frac{i}{8}\bar{u}_1=0.\qquad(6.3.16)$$

Now recall that after the exotic twist, on the worldsheet u_1 was a section of K and u_2 of $\mathbb{C}$. Since $\bar{\partial}=\frac{\partial}{\partial\bar{w}}$ should be interpreted as a $(0,1)$-form on Σ, it is a section of $\overline{K}$. Therefore, $\bar{\partial}u_1$ is a section of $\mathbb{C}$, while $\bar{\partial}u_2$ is a section of $\overline{K}$. It follows that the flow equation makes sense globally on Σ, as all terms sit in the same bundle over Σ. Note that on our choice of $\Sigma=\mathcal{C}$, the cylinder, this equation is actually always well-defined, since all bundles are trivial, so defined globally, on $\mathcal{C}$.

Imposing boundary conditions

Now we only need to specify the boundary condition $\mathcal{A}$ in (6.3.1). This is determined by the condition that at $s = -\infty$, the cylinder $\mathcal{C}$ is mapped into a middle-dimensional subset V of a critical subset of h. For the SHO, it turns out that there is only 1 critical subset, in general, there could by multiple. The critical subset can conveniently be described as all periodic solutions of the complexified Hamilton equations (6.2.6) on $\mathbb{C}^2$. Equivalently, one can look at the stationary points of (6.2.10) (where $\frac{\partial \xi}{\partial s} = 0$) to find the same critical subset.

For the simple harmonic oscillator, a generic periodic solution corresponds just to a product of two circles, each in a copy of $\mathbb{C}$, spanned by the I-holomorphic coordinates u_1 or u_2. This choice is consistent with the flow equation (6.2.10), that uses complex structure I. The set of all such circles is labeled by two radii in $\mathbb{R}_+ \times \mathbb{R}_+$ and forms the critical subset of h. We can now choose V by flowing from either the set of circles in the u_1-plane or the u_2-plane. For instance, choosing the latter, these are solutions to

$$\dot{\xi}^4 = \xi^2, \qquad \dot{\xi}^2 = -\xi^4,$$

so V simply becomes the space $V = \{\xi \mid \forall t \in S^1 : \xi(t) \in \mathcal{L} \ \& \ (\xi^2)^2 + (\xi^4)^2 = E^2, E \in \mathbb{R}_+\}$ of circles that lie in the Lagrangian subspace $\mathcal{L} = \{u_1 = 0\}$.

This is a choice we have to manually enforce, as for path integrals involving open worldsheets one always has to prescribe a suitable boundary condition $\mathcal{A}$ in the path integral. An insertion of a non-local operator can only alter these boundary conditions.

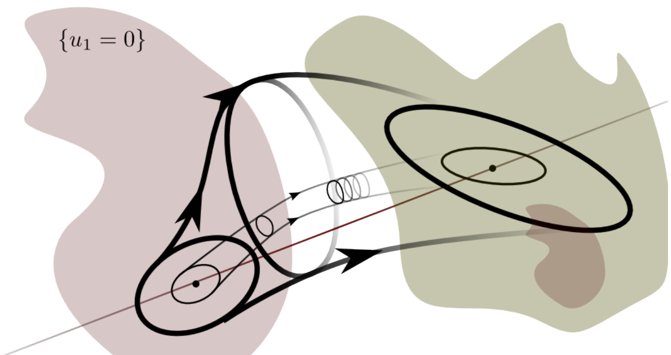


Figure 8: Exotic integration cycle in the free loop space $L\mathcal{M}_{\mathbb{C}}$ for the simple harmonic oscillator.

Within the above construction, we recall (6.3.1) that gives the partition function Z_{SHO} of the harmonic oscillator. Upon localization the only contribution comes from maps Φ that have $S_{A,LG} = 0$, obey the flow equation (6.3.16) and the boundary condition

$$\Phi(s = -\infty, t) \in V. \tag{6.3.17}$$

The A-model path integral (6.3.1) thus reduces to the boundary path integral

$$Z_{\text{SHO}} = n_\Phi(\mathcal{A}) \int_{\mathcal{C}_V \subset L\mathcal{M}_{\mathbb{C}}} \mathcal{D}\phi \exp\left(i \oint_{\partial\mathcal{C}} \left(B_\mu d\phi^\mu - \mathcal{H} dt \right) \right). \tag{6.3.18}$$

Here $\mathcal{C}_V$ represents the exotic integration cycle: a collection of loops in $\mathcal{M}_{\mathbb{C}}$ obtained by downward Morse flow from concentric circles in $\{u_1 = 0\}$. Moreover, $B_\mu \phi^\mu$ is the complex form of $\Lambda_A d\xi^A$ and $n_\Phi(\mathcal{A}) = 1$ is the open Gromov-Witten invariant: the number of inequivalent embeddings of $\mathcal{C}$ that satisfy (6.3.17), which is just 1. This follows from the simple-connectedness of $\mathcal{M}_{\mathbb{C}} = \mathbb{C}^2$.

Finally, recall that ordinary quantization of $\mathcal{M}$ for the SHO Hamiltonian gives the space of states sitting in a representation of the oscillator algebra:

$$\mathcal{H} = \left\{ |n\rangle \mid E_n = (n + \frac{1}{2})\hbar, n \in \mathbb{Z}_{\geq 0} \right\}, \tag{6.3.19}$$

where we restored $\hbar$. Its trace coincides with that of $\mathrm{Hom}(\mathcal{B}_{\mathcal{L}}, \mathcal{B}_{cc})$, with our earlier described choices for the Lagrangian A-brane supp $\mathcal{B}_{\mathcal{L}} = \{u_1 = 0\}$ and $\mathcal{B}_{cc}$ with gauge field curvature ω_I. As a natural generalization of the A-model quantization to $H \neq 0$, an interesting relation between the two spaces is revealed: it seems most probable that these spaces should be isomorphic.

7

Chern-Simons theory

In chapter 3 we saw how we could twist field theories to make them topological. In this section we discuss a second type of topological theory: theories with a manifestly metric-invariant Lagrangian. In this fashion we obtain a classically manifestly topological theory of *Schwarz-type*. However, showing that such a theory remains topological at the quantum level is usually non-trivial. The canonical example is *Chern-Simons theory*, which is a *topological gauge theory* and central in the study of topological field theory and knot theory.

For us, the most important property of Chern-Simons theory is that a full non-perturbative description exists: the theory is fully solvable. We will show that Chern-Simons theory exactly computes polynomial knot invariants, such as the *Jones polynomial*. We then finish with a discussion of the generalization of the Jones polynomial by its categorification, *Khovanov homology*. Categorification here just means that Khovanov homology assigns vector spaces, instead of numbers, to a knot. Khovanov homology will be the motivation to pursue the duality in chapter 8.

7.1 Basics

Let G be a compact semisimple Lie group and suppose we have a principal G-bundle $E \longrightarrow M$ on an oriented closed 3-manifold M with gauge field A, which is an $\mathfrak{g}$-valued 1-form in $\Omega^1(M, \mathrm{ad}\, E)$. Normally, we could try to define a Yang-Mills type Lagrangian to describe the dynamics. However, in any odd number of dimensions there is an alternative choice: the Chern-Simons action. In 3 dimensions, the Chern-Simons action is given by

$$\mathcal{S}_{CS}(A) = \frac{k}{4\pi}\int_M \mathrm{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) \equiv \frac{k}{4\pi}\int_M \mathrm{CS}(A) \tag{7.1.1}$$

The *level* $k > 0$ plays the role of inverse coupling constant of the theory and tr is a suitably normalized quadratic form on the Lie algebra $\mathfrak{g}$ of G. Note that reversing the sign of k implies changing the orientation on M. For the fundamental representation of $G = SU(N)$, tr is just the ordinary matrix trace. We shall discuss below how we determine the normalization of tr. If M is oriented and G is simply connected and compact, every principal G-bundle on M is trivializable (always admits global sections). This makes the above expression well-defined. However, if G is not simply connected, there can be non-trivial principal G-bundles and we should sum over all possible bundles, see [18]. Note that the Lagrangian $\mathrm{CS}(A)$ is manifestly metric-independent, as the integration of top differential forms does not require a metric. Therefore, the theory is classically topological: the classical action is independent of the choice of a metric on M.

The most natural way to understand the role of the Chern-Simons Lagrangian uses Stokes' theorem: any 3-manifold M can be thought of as the boundary of some 4-manifold V. If it is possible to extend the principal G-bundle $E \longrightarrow M$ to a bundle $E' \longrightarrow V$ we can relate the Chern-Simons action on the former with topological information on the latter. The easiest example is $V = M \times [0,1]$, using the coordinate $t \in [0,1]$. In this case, V retracts onto M, which makes the extension $E' \longrightarrow M$ unique.
By the Poincaré lemma we locally have $\mathrm{tr}\,(F\wedge F) = d\,\mathrm{tr}\,(\mathrm{CS}(A))$, leaving the wedges implicit:

$$d\,\mathrm{tr}\left(AdA + \frac{2}{3}A^3\right) = \mathrm{tr}\left(dA^2 + \frac{2}{3}\left(dAA^2 - A(dA)A + A^2dA\right)\right) = \mathrm{tr}\left(dA^2 + 2dAA^2\right),$$

$$\mathrm{tr}\,(F\wedge F) = \mathrm{tr}\left((dA + A^2)^2\right) = \mathrm{tr}\left(dA^2 + dAA^2 + A^2dA + A^4\right) = \mathrm{tr}\left(dA^2 + 2dAA^2\right).$$

From this we can conclude that $\int_M \mathrm{tr}\,\mathrm{CS}(A) = \int_V \mathrm{tr}\, F \wedge F$, when an extension $E' \longrightarrow V$ exists. If we have two different connections A, A' on the G-bundle $E \longrightarrow M$, then choosing the connection A at $t = 0$ and A' at $t = 1$ in the bundle over $M \times [0, 1]$, by Stokes theorem we get

$$\mathcal{S}_{CS}(A) - \mathcal{S}_{CS}(A') = \frac{k}{4\pi} \int_{M\times[0,1]} \mathrm{tr}\,(F \wedge F)\,. \tag{7.1.2}$$

This relates two inequivalent connections to each other: their action differs by an integer multiple. This correspondence can be generalized to any V over which we can globally extend our principal G-bundle, there is a further analysis of these issues in [18]. So we see that the Chern-Simons theory on the boundary M captures topological information of the associated 'bulk' theory on V.

The partition function, also known as the Witten-Reshetihkin-Turaev invariant, for this theory is given by the path integral:

$$\begin{aligned} Z_{CS}(M) &= \frac{1}{\mathrm{Vol}(G)} \int_{\mathcal{A}} \mathcal{D}A \exp\left(i\mathcal{S}_{CS}(A)\right) = \frac{1}{\mathrm{Vol}(G)} \int_{\mathcal{A}} \mathcal{D}A \exp\left(ik \int_M \mathrm{CS}(A)\right) \\ &= \frac{1}{\mathrm{Vol}(G)} \int_{\mathcal{A}} \mathcal{D}A \exp\left(\frac{ik}{4\pi} \int_M \mathrm{tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)\right). \end{aligned} \tag{7.1.3}$$

Here we integrate over the space $\mathcal{A}$ of gauge connections on the principal G-bundle. Note that G acts on the space of connections by gauge transformations $A \longmapsto gAg^{-1} + dgg^{-1}$, so we should integrate over the conjugation classes in the space of connections. In perturbation theory this requires, in standard practice, the introduction of Fadeev-Popov ghosts to fix the gauge, upon which the volume factor in front is cancelled. Note that this is a formal expression for non-compact G, because then the volume of G is infinite: this is circumvented by dividing any correlator by $Z(M)$.

To be a consistent theory, the integrand in the path integral (7.1.3) should be single-valued. Under a gauge transformation $A \longmapsto A' = gAg^{-1} - dgg^{-1}$ where g is an element of a simply connected compact gauge group G, the Chern-Simons functional is shifted $\int_M \mathrm{CS}(A) \longmapsto \int_M \mathrm{CS}(A) + 8\pi^2$, where we now pick the normalization of the trace to ensure that the shift is exactly $8\pi^2$. Indeed we have:

$$\begin{aligned} &\mathrm{tr}\left(A' \wedge dA' + \frac{2}{3} A' \wedge A' \wedge A'\right) \\ &= \mathrm{tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) - d\,\mathrm{tr}\left(gA \wedge d(g^{-1})\right) + \frac{1}{3}\,\mathrm{tr}\left(dgg^{-1} \wedge dgg^{-1} \wedge dgg^{-1}\right), \end{aligned} \tag{7.1.4}$$

which is straightforward to check (D.1). Now the second term is a total derivative, so if M is closed, that term will vanish upon integration over M by Stokes theorem. The third term does not vanish, even if M is closed, and is a topological invariant of the gauge transformation g.

This follows from the fact that for a semisimple compact Lie group G, the third homotopy group is isomorphic to the integers: $\pi_3(G) = \mathbb{Z}$, which reflects that there are *large gauge transformations*, i.e. gauge transformations that are not smoothly homotopic to the identity transformation.* Hence any gauge field or gauge transformation is classified by its winding number n, computable from

$$2\pi n = \frac{1}{12\pi} \int_M \mathrm{tr}\left(dgg^{-1} \wedge dgg^{-1} \wedge dgg^{-1}\right) \in 2\pi\mathbb{Z}. \tag{7.1.5}$$

Hence we see that the Chern-Simons functional transforms under large gauge transformations as

$$\frac{k}{4\pi} \int_M \mathrm{CS}(A) \overset{G}{\longmapsto} \frac{k}{4\pi} \int_M \mathrm{CS}(A) + 2\pi nk. \tag{7.1.6}$$

*As an example, consider Chern-Simons theory with $G = SU(2)$ on S^3. A gauge transformation is generated by a local element $g(x)$ of the gauge group $SU(2)$: g is a map $g : S^3 \longrightarrow SU(2)$. Not all such maps are contractible to the identity map, because $SU(2) \cong S^3$: since $\pi_3(S^3) = \mathbb{Z}$. Note that from this isomorphism it follows that $\pi_3(SU(2)) \cong \pi_3(S^3)$. It is a nontrivial result that $\pi_3(G) = \mathbb{Z}$ for semisimple compact G.

Since n is in general arbitrary, for single-valuedness of the integrand in the path integral (7.1.3) the level k should be an integer. Note that this is a quantum constraint: the classical physics is entirely independent of the choice of level k. Note that this result holds only for compact gauge groups G: for non-compact G, the third homotopy class is not necessarily isomorphic to $\mathbb{Z}$: as an example, for the non-compact Lie group $SL(2,\mathbb{R})$ we have $\pi_3(SL(2,\mathbb{R})) = 0$. In that case, we do not have a topological obstruction on k.

The Euler-Lagrange equations for A follow from extremizing the action.

$$\delta\mathcal{S}_{CS} = \delta\left(\frac{k}{4\pi}\int_M \mathrm{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)\right) = 2\frac{k}{4\pi}\int_M \mathrm{tr}\left((dA + A\wedge A)\wedge\delta A\right) = 0 \quad (7.1.7)$$

from which we find (see (D.1)) the equation of motion

$$dA + A\wedge A = F_A = 0. \quad (7.1.8)$$

So classical solutions correspond to flat connections. Note that this equation is independent of the level k, hence the classical physics independent of the level k. We also see that, in our earlier notation, if a flat connection on M extends to a flat connection on V by our correspondence (??) we see immediately that such a flat connection has $\mathcal{S}_{CS}(A) = 0$.

Flatness is preserved under gauge transformations since $F \overset{G}{\longmapsto} gFg^{-1}$ (section 2). Hence there is a special class of flat connections, namely the gauge orbit of the trivial connection $A = 0$ which consists of *pure gauge* connections of the form $-dgg^{-1}$. Note that on spaces with non-trivial topologies, in general these are not all flat connections. This is because the flatness equation $F = 0$ is a *local* statement. In particular, this means that on a infinitesimal loop, the holonomy of a flat connection is trivial, and more generally one can show that its holonomy is trivial on any contractible loop: flatness is *homotopically preserved.*‖ However, on non-contractible loops, flat connections may have non-trivial holonomy. Since we can concatenate loops, it's clear that a flat connection is determined by its holonomies around non-trivial loops, phrased more precisely: every gauge equivalence class of a flat connections is in exact correspondence with a homomorphism $v : \pi_1(M) \longrightarrow G$. We denote the space of all such homomorphisms as $\mathrm{Hom}(\pi_1(M), G)$.

From the above considerations it follows that the appropriate phase space of classical solutions is the space of gauge equivalence classes of flat connections, also called the *moduli space of flat connections*

$$\mathcal{M}^G = \{A \in \mathcal{A} \mid F_A = 0\}/G = \mathrm{Hom}(\pi_1(M), G)/G. \quad (7.1.9)$$

where we quotient out by gauge transformations. $\mathcal{M}^G$ has a very intricate topology in general, which is one of the reasons that explicit calculations in Chern-Simons are a-priori not straightforward.

Observables

In Chern-Simons theory, observables are furnished by *Wilson loops* which compute the holonomy of the gauge field A around closed curves on M: they capture only global information of M. A Wilson loop W is defined as the unique solution to the parallel transport equation: suppose we are given a path C from $x_0 \in M$ to $x \in M$, then a basis element $e_i(x_0)$ of the fiber at x_0 is mapped to another basis element $e_i(x) = g_i^{\ j} e_j(x_0)$ of the fiber at x by an element $g(x) \in G$. Any vector $\psi = \psi_i e_i$ then satisfies the equation for parallel transport at any x:

$$D\psi = 0 \Longrightarrow Dg = dg + Ag = 0. \quad (7.1.10)$$

If the gauge group G is abelian, we can simply view this as the defining equation of the (matrix) exponential and write the solution as $g(x) = \exp\int_{x_0}^{x} A \in G$. Then we can make a scalar operator out of this by taking a trace

$$\mathrm{tr}\exp\int_C A. \quad (7.1.11)$$

‖The proof of this follows easily by dividing the area enclosed by the contractible loop into a grid of infinitesimal loops: all internal contributions will cancel, leaving only the edges giving non-zero contributions.

This expression has a natural physical interpretation: if we imagine a particle that has unit electrical charge under the $U(1)$ symmetry of electromagnetism, then the Wilson loop amounts exactly to a contribution to the action of $q\int_C A$, which is exactly the addition to the action due to a charged particle traveling in the electromagnetic potential A.

However, when G is not abelian, there will be ordering ambiguities and we cannot simply exponentiate to get a solution of equation (7.1.10). However, we may denote the unique solution to equation (7.1.10) as the formal expression

$$W_C(A) = \operatorname{tr} \mathrm{P}\exp\left(\int_C A\right) \tag{7.1.12}$$

where we denoted by $\mathrm{P}\exp$ the path ordered exponential. One should think of this as a ordering prescription on the products of gauge fields that appear in the exponential, upon which ultimately physical relevant quantities like correlation functions are not dependent in the end**.

Perturbative issues

Firstly, we saw that the Chern-Simons action was metric-independent, which makes the theory topological classically. Now the calculation of correlators of Wilson loops can be done perturbatively: one can show that the metric-independence continues to hold at the perturbative level, by showing that perturbative quantum calculations give almost topological invariants, which are metric independent. They are not truly 'topological' as one does need to choose a framing: a choice of local trivializations of TM and the knots. This issue is covered in more detail in [19].

7.2 Canonical quantization of Chern-Simons theory

We now come to the magical property of Chern-Simons theory: it has a full non-perturbative solution. The idea is to cut M in pieces and canonically quantize the theory on each piece. After quantization, one can glue the pieces back together using the axiomatic rules of topological field theory to obtain all correlation functions for Chern-Simons theory on M. These rules are elegantly captured by category theory, which we discuss in appendix A.4.

Here, our spacetime M can be non-compact, but we will see that we can get the most direct answers when M is compact. By an indirect argument, we can then extend our answers to the case of non-compact M.

Given a 3-manifold M, we can always cut it along a Riemann surface Σ, around which M locally looks like $\Sigma \times \mathbb{R}$ when no Wilson loops are present on M. We can interpret $\mathbb{R}$ as the time direction and Σ as an initial value surface. If Wilson loops are present, Σ will contain punctures. We shall first describe canonical quantization on $\Sigma \times \mathbb{R}$ without Wilson loops.

The principal G-bundle can be restricted to $E \longrightarrow \Sigma \times \mathbb{R}$ and for the gauge field A on this bundle we identify the fields A_i as canonical coordinates and their time derivatives $\partial_t A_i$ as canonical momenta. The Chern-Simons action in index notation is:

$$\mathcal{S}_{CS}(A) = \frac{k}{4\pi}\int_{\Sigma\times\mathbb{R}} d^2x dt \epsilon^{ijk} \operatorname{tr}\left(A_i\partial_j A_k + \frac{2}{3}A_iA_jA_k\right) \tag{7.2.1}$$

where $(x,t) \in \Sigma \times \mathbb{R}$ and ϵ is the total antisymmetric tensor. On $\Sigma \times \mathbb{R}$, we can decompose the exterior derivative as $d = dt\frac{\partial}{\partial t} + \tilde{d}$ and the gauge field as $A = A_0 + \tilde{A}$, upon which the Lagrangian becomes

$$\mathcal{L} = \frac{k}{4\pi}\int dt \int_\Sigma d^2y \tilde{\epsilon}^{ij} \operatorname{tr}\left(\tilde{A}_i\partial_t\tilde{A}_j\right) + \frac{k}{2\pi}\int_{\mathbb{R}} dt \int_\Sigma d^2y \operatorname{tr}\left(A_0\left(\tilde{d}\tilde{A} + \tilde{A}^2\right)\right) \tag{7.2.2}$$

**This is also clear from considering a solution to the holonomy equation (7.1.10) as follows: for any parametrized path γ, consider a small segment $\delta\gamma = \frac{d\gamma}{dt}\delta t$ on which we do parallel transport using the connection A. Then we can obtain the holonomy along the curve γ as $g = \lim_{\delta t\longrightarrow 0}\left[\exp\left(A\left(\frac{d\gamma}{dt}(0)\right)\right)\exp\left(A\left(\frac{d\gamma}{dt}(\delta t)\right)\right)\exp\left(A\left(\frac{d\gamma}{dt}(2\delta t)\right)\right)\ldots\right]$ which we cannot write as a single exponential due to the non-abelianity of G. We rather summarize this expression as the formal path-ordering exponential $\mathrm{P}\exp\int_\gamma A$.

where $\tilde{\epsilon}^{ij}$ is now the total antisymmetric tensor in the space directions. Note that A_0 has no canonical momentum: there is no $\partial_t A_0$ term in the Lagrangian: it is not a dynamical field, instead A_0 acts as a Lagrange multiplier. It is a multiplier for a 'Gauss law', since as seen above, A_0 multiplies $\tilde{\epsilon}^{ij}\tilde{F}_{ij}$, the space part of the curvature of A. To identify the space of physical states we constrain first. So we first restrict ourselves to flat connections and take into account gauge equivalence: gauge equivalent connections have the same action as shown before. Hence the phase space of Chern-Simons theory on $\Sigma \times \mathbb{R}$ is exactly the moduli space of flat connections $\mathcal{M}_\Sigma^G$ on the restricted bundle $E' \longrightarrow \Sigma$. It turns out that this space is compact [19] and hence we deduce that the space of physical Chern-Simons states will be finite-dimensional: its dimension is most easily seen from considering $\mathrm{Hom}(\pi_1(M), G)$: $\dim(\pi_1(M)) = 2g - 2$ since $\pi_1(M)$ has $2g$ generators and obeys 1 relation $\prod aba^{-1}b^{-1} = 1$ and we need to mod out by conjugacy, so from plain linear algebra we get $\dim \mathcal{M}_{\Sigma_g}^G = (2g-2)\dim G$.

Studying this constraint is the key to the non-perturbative description of Chern-Simons theory.

The relation to the Wess-Zumino-Witten CFT

The classical constraint was that we should restrict ourselves to flat connections, for which $\tilde{\epsilon}\tilde{F} = 0$. With r marked points (sources) on Σ, this relation is modified to

$$\frac{k}{4\pi}\tilde{\epsilon}^{ij}\tilde{F}_{ij}^a(x) = \sum_{i=1}^{r} \delta^{(2)}(x - p_i)T_i^a. \tag{7.2.3}$$

Suppose now that we first quantized and afterwards imposed the flatness constraint: in that case we would have an operator constraint on wavefunctions: choosing $A_{\bar{z}}$ as our canonical coordinate, any physical Chern-Simons wavefunction $\Psi_{CS}[A_{\bar{z}}]$ should obey:

$$\frac{k}{4\pi}F_{\bar{z}z}^a\Psi_{CS}[A_{\bar{z}}] = \sum_{i=1}^{r} \delta^{(2)}(x - p_i)T_i^a\Psi_{CS}[A_{\bar{z}}]. \tag{7.2.4}$$

In terms of the *operators* $A_{\bar{z}}$ and $\frac{2\pi}{k}\frac{\delta}{\delta A_{\bar{z}}}$, the curvature operator reads

$$F_{\bar{z}z} = \partial_{\bar{z}}A_z - \partial_z A_{\bar{z}} + [A_{\bar{z}}, A_z] = -\partial_z A_{\bar{z}} + \frac{2\pi}{k}D_{\bar{z}}\frac{\delta}{\delta A_{\bar{z}}}. \tag{7.2.5}$$

where we used the gauge covariant derivative $D_{\bar{z}} = \partial_{\bar{z}} + [A_{\bar{z}}, .]$.

Without knots

Suppose there are no knot insertions on M, then the flatness constraint reduces to

$$\begin{aligned} F_{\bar{z}z}^a\Psi_{CS}[A_{\bar{z}}] &= \left(\frac{2\pi}{k}\partial_{\bar{z}}\frac{\delta}{\delta A_{\bar{z}}^a} - \partial_z A_{\bar{z}}^a + \frac{2\pi}{k}\left[A_{\bar{z}}, \frac{\delta}{\delta A_{\bar{z}}}\right]^a\right)\Psi_{CS}[A_{\bar{z}}] \\ &= \left(-\frac{k}{2\pi}\partial_z A_{\bar{z}}^a + D_{\bar{z}}\frac{\delta}{\delta A_{\bar{z}}^a}\right)\Psi_{CS}[A_{\bar{z}}] = 0, \end{aligned} \tag{7.2.6}$$

which is equivalent to

$$\left(\delta^{ac}\partial_{\bar{z}} + f^{abc}A_{\bar{z}}^b(z)\right)\frac{\delta}{\delta A_{\bar{z}}^c(z)}\Psi_{CS}[A_{\bar{z}}] = \frac{k}{2\pi}\partial_z A_{\bar{z}}^a(z)\Psi_{CS}[A_{\bar{z}}]. \tag{7.2.7}$$

The Chern-Simons Hilbert space $\mathcal{H}_{CS}$ corresponds to normalizable solutions to these equations. We want to prove the identification:

$$\mathcal{H}_{CS}(\Sigma \times \mathbb{R}) \cong \{\text{conformal blocks of WZW theory on } \Sigma\}\,.$$

To show this, the idea is to couple the WZW currents J to a background gauge field A, upon which we can write down a wavefunction

$$\Psi_{WZW}[A_{\bar{z}}] = \langle \exp\left(-\frac{1}{\pi}\int A_{\bar{z}}J\right)\rangle. \tag{7.2.8}$$

By calculation, one shows that the flatness constraint (7.2.7) with $\Psi_{WZW}[A_{\bar{z}}]$ is equivalent to the JJ operator product expansion

$$J^a(z)J^b(w) = \frac{k\delta^{ab}}{(z-w)^2} + f^{ab}{}_c \frac{J^c(w)}{(z-w)} + \dots . \tag{7.2.9}$$

Since the JJ OPE is equivalent to the WZW Ward identities, which uniquely characterize the conformal blocks of WZW theory, one has the wanted identification. The relevant calculation can be found in appendix (D.2).

With knots

This identification continues to hold when we have Wilson loops on M: now we have to do quantization on $\Sigma - \{p_i\}_i$, where the p_i represent punctures where the Wilson lines cross Σ. Flat connections will now be classified by their holonomy around these punctures, that is, the flatness constraint becomes

$$\frac{k}{4\pi} F^a_{\bar{z}z} = \sum_{i=1}^{n} \delta(z-p_i) T^a_i . \tag{7.2.10}$$

Here the $T^a_{(i)}$ are the generators associated to the representation R_i at p_i, a is a group index. Now quantization of this equation is not straightforward: the connection that satisfies this constraint cannot be quantized to a scalar operator, since the generators $T^a_{(i)}$ are not commuting for non-abelian gauge groups.

We can circumvent the problematic operators T^a, by employing the Borel-Weil-Bott theorem. This result enables us to exchange the quantum problem for a classical one. This works as follows. One introduces the *flag manifold* G/T where T is the maximal torus in G and for every representation R_i we introduce a symplectic structure ω_{R_i} on G/T. This symplectic structure is such that quantization of the classical phase space $(G/T, \omega_{R_i})$ gives back R_i. This allows us to replace the operators $T^a_{(i)}$ on the right-hand side by *functions* on G/T that would quantize to the $T^a_{(i)}$. It turns out that this function is exactly a moment map μ.

Intuitively, it should be clear what the punctures on Σ correspond to: marked points should correspond to primary operator insertions on Σ, which transform in some representation R_i of the gauge group G. Then the space of physical states $\mathcal{H}_{\Sigma,p_i,R_i}$ will correspond to the space of conformal blocks associated with the correlation functions of those primary operators. One can show that indeed the generalized flatness constraint (7.2.10) is equivalent to the *Knizhnik-Zamolodchikov equations*

$$\partial_{z_j}\psi(z_1,\dots,z_n) = \frac{1}{k+h}\sum_{a,p\neq q} \frac{T^a_p \otimes T^a_q}{z-z_p}\psi(z_1,\dots,z_n). \tag{7.2.11}$$

These equations uniquely determine the WZW conformal blocks: they are defining differential equations for correlation functions of Virasoro primary fields. This correspondence is worked out for Chern-Simons on S^3 in [20]. In this way we again conclude that Chern-Simons physical states correspond to WZW conformal blocks.

At this point, the nature of the representation of the operator insertion at the marked point comes into play: if the representation is complex, then the knot should be oriented since complex conjugation interchanges inequivalent complex representations. If the representation is real, then complex conjugation does nothing and the knot can be unoriented. This indicates already that for G with real representations, any relation we can write down is less constrained than in the real case.

7.3 Chern-Simons theory and knot polynomials

Using the identification above, we can now compute Chern-Simons correlation functions using the WZW CFT. First, we consider closed M. Then it is a fact from topology that any such manifold can be obtained from S^3 by *surgery*. Surgery leads to recursion rules, which allow us to express the correlation function for any link on any closed M in terms of a correlator on S^3 with extra links included. It turns out that we only need to consider the surgery operations in the case of genus 0 and 1, since any surgery can be obtained from the genus 1 case, the genus 0 fixes the overall normalization. The intuition for this is that the genus 1 surgery allows us to remove holes, which can be iterated as needed. Effectively, this means that we only need to describe the canonical quantization of Chern-Simons theory on $\Sigma \times \mathbb{R}$ where Σ is of genus 0 or 1.

- In genus 0 without punctures, the physical Hilbert space is the 1-dimensional space of conformal blocks of WZW theory with gauge group G at level k. With punctures, we already derived a selection rule for allowable representations of the primary operator insertions. This enables us to describe knots on S^3.
- Canonical quantization associates to the torus $\Sigma = T^2$ the Hilbert space $\mathcal{H}(T^2)$ of WZW conformal blocks on T^2, which are in one to one correspondence with the integrable representation of the affine Lie algebra associated to G at level k.

The canonical reference for all CFT related material is [21].

At genus 0

Suppose Σ has genus 0. Since the theory has to be gauge-invariant, if punctures are present on Σ the representations and charges associated with the punctures should always be such that all charges sum to zero, since at weak coupling (large k) all charges are 'static' and are decoupled from the gauge field. For finite k the physical Hilbert space will be a subspace of the G-invariant subspace $\mathcal{H} = \mathrm{Inv}(\otimes_i R_i)$. Hence, we can list the possibilities for a low number of punctures r and for the case where R is a complex fundamental representation of G:

- $r = 0$: $\mathcal{H}$ has dimension 1. This corresponds to the fact that in a CFT, for descendants of the identity there is only one conformal block.
- $r = 1$: $\dim \mathcal{H}_{p,R} = 1$ if $R = \mathbf{1}$, the trivial representation and $\dim \mathcal{H} = 0$ otherwise.
- $r = 2$: $\dim \mathcal{H}_{p_1,p_2,R_1,R_2} = 1$ if $R_1 = \overline{R}_2$, and is 0 otherwise.
- $r = 3$: In this case $\dim \mathcal{H}_{p_i,p_j,p_k,R_i,R_j,R_k} = N_{ijk}$, where N_{ijk} are the coefficients in the *Verlinde fusion algebra*, $\phi_i \times \phi_j = N_{ij}^k \phi_k$, where the labels label highest weight representations. We shall not really use this here.
- $r > 3$. If we know the coefficients N_{ijk}, we can deduce the dimensions of $\mathcal{H}$ for all higher number of punctures. From the fusion algebra, one can recursively express all cases in terms of N_{ijk}. One special case will be especially important to us: $r = 4$ and representations $R, R, \overline{R}, \overline{R}$, for which we have
$$R \otimes R = \bigotimes_{i=1}^{s} E_i \tag{7.3.1}$$
with E_i distinct irreducible representations of the gauge group G. $R \otimes R \otimes \overline{R} \otimes \overline{R}$ is uniquely fixed by the decomposition of $R \otimes R$, by complex conjugation. So $\dim \mathcal{H} = s$, at large k (the weak coupling limit). As an example, for $SU(3)$, we have $\mathbf{3} \otimes \mathbf{3} = \mathbf{3} \oplus \mathbf{6}$, the decomposition into the antisymmetric $\mathbf{3}$ and the symmetric $\mathbf{6}$. In general, $s = 2$ for $SU(N)$, so $\dim \mathcal{H}_{p_1,p_2,p_3,p_4,R,R,\overline{R},\overline{R}} = 2$, and only for $k = 1$ it is 1-dimensional.

If the representation R is real, then the most important modification for us comes about in the case for four punctures. For concreteness, let us take $G = SO(N)$. Then any tensor product of two real fundamental representations can be decomposed as
$$R \otimes R = S \oplus A \oplus \mathbf{1} \tag{7.3.2}$$
where S is the symmetric representation, A is the antisymmetric representation and $\mathbf{1}$ is the trivial one. We see that $s = 3$ in for $SO(N)$.

Cutting and pasting on M uses the discussion from section A.4. Chern-Simons theory assigns to a closed manifold the partition function $Z(M)$. Moreover, it assigns to a manifold with boundary Σ a Hilbert space $\mathcal{H}(\Sigma)$, which depends on a choice of orientation of Σ: once this is picked, the topological field theory should assign the dual space $\mathcal{H}^*$ to Σ with the opposite orientation, which we denote by $-\Sigma$.

We first restrict to compact M. Slicing M along a Riemann surface $\Sigma = S^2$ of genus 0 splits $M = M_- \cup M_0$. We take the convention such that M_- is bounded by $-\Sigma$ with the negative orientation and M_0 by Σ with the positive orientation. Then our topological field theory assigns a vector $\chi \in \mathcal{H}^*$ to M_0 and a vector $\chi' \in \mathcal{H}$ to M_-, for which we have

$$(\chi, \chi') = Z(M). \tag{7.3.3}$$

Note that by our discussion in the previous section, $\mathcal{H}$ and $\mathcal{H}^*$ are both 1-dimensional. Now consider $S^3 = S^3_- \cup S^3_+$, sliced through the equator S^2. Then the same reasoning gives us a vector $v' \in \mathcal{H}_{S^2}$ and a vector $v \in \mathcal{H}^*_{S^2}$ such that $(v, v') = Z(S^3)$, where again the Hilbert spaces are 1-dimensional. But the Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_{S^2}$ should be the same, so v is a multiple of χ and v' is a multiple of χ'. Hence

$$(\chi, \chi')(v, v') = (\chi, v')(v, \chi') \Longrightarrow Z(M)Z(S^3) = Z(M_1)Z(M_2) \tag{7.3.4}$$

where $M_1 = M_- \cup S^3_+$ and $M_2 = M_0 \cup S^3_-$. Normalizing this equation, we learn that

$$\frac{Z(M)}{Z(S^3)} = \frac{Z(M_1)}{Z(S^3)} \frac{Z(M_2)}{Z(S^3)}. \tag{7.3.5}$$

Now consider the special case that we have N unlinked circles or unknots on S^3. Then we can repeatedly cut S^3 in such a way that we do not intersect the unknots, so that repeated use of the above procedure tells us that

$$\frac{Z(S^3; C_1, \ldots, C_N)}{Z(S^3)} = \prod_{i=1}^{N} \frac{Z(S^3, C_i)}{Z(S^3)}. \tag{7.3.6}$$

For later use, we define the correlation function

$$\langle C_1, \ldots, C_N \rangle \equiv \frac{Z(S^3; C_1, \ldots, C_N)}{Z(S^3)}. \tag{7.3.7}$$

Note that up to now, we didn't need the explicit value of the 3-manifold invariant $Z(S^3)$. We will give an explicit expression for $Z(S^3)$ in section ??.

At genus 1

Consider two manifolds M_-, M_0 with at least one common boundary Σ, which we take to have opposite orientations on $M_\pm$. For definiteness, we assume that the orientation is such that M_0 is assigned $\mathcal{H}(\Sigma)$ and M_- is assigned $\mathcal{H}^*(\Sigma)$. The TFT path integral on $M_\pm$ computes states in these Hilbert spaces

$$\langle \Psi_{M_-} | \in \mathcal{H}^*(\Sigma), \qquad |\Psi_{M_0}\rangle \in \mathcal{H}(\Sigma). \tag{7.3.8}$$

As before, we can glue M_-, M_0 along Σ using a diffeomorphism $f : \Sigma \longrightarrow \Sigma$ which is represented by a (unitary) operator $U(f)$ that acts on the Hilbert space, $U(f) : \mathcal{H}(\Sigma) \longrightarrow \mathcal{H}(\sigma)$, so the partition of M is

$$Z(M) = \langle \Psi_{M_-} | U(f) | \Psi_{M_0} \rangle. \tag{7.3.9}$$

In the case that $\Sigma = T^2$ there are special diffeomorphisms: namely those generated by the modular group $SL(2, \mathbb{Z})$ that acts on T^2. This follows from a result on the mapping class group of T^2: we have $\mathrm{Diff}/\mathrm{Diff}_0(T^2) = SL(2, \mathbb{Z})$: all 'large' diffeomorphisms, the ones that cannot be obtained by exponentiation of infinitesimal diffeomorphisms, are exactly generated by $SL(2, \mathbb{Z})$. This also implies that any diffeomorphism on T^2 can be obtained by a combination of a 'small' diffeomorphism from $\mathrm{Diff}_0(T^2)$ and 'large' diffeomorphism from $SL(2, \mathbb{Z})$. In a basis where the two cycles of T^2 are the meridian $m = (1, 0)$ and the longitude $l = (0, 1)$ (going the '*l*ong' way around the torus), the generators S and T of $SL(2, \mathbb{Z})$ are represented by the matrices

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{7.3.10}$$

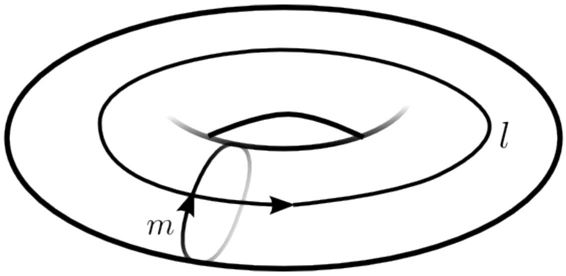


Figure 9: The torus T^2 with the basis l, m of 1-cycles.

The generator T generates the Dehn twist, whereas S exchanges the m and l cycle. These generators then can be lifted to operators on $\mathcal{H}(\Sigma)$ as

$$T_{RR'} = T_{pp'} = \delta_{pp'} \exp 2\pi i \left(h_p - c/24\right), \tag{7.3.11}$$

$$S_{RR'} = S_{pp'} = \frac{i^{|\Delta_+|}}{(k+h)^{r/2}} \left(\frac{\mathrm{Vol}\,\Lambda^w}{\mathrm{Vol}\,\Lambda^r}\right)^{\frac{1}{2}} \sum_{w \in \mathcal{W}} \epsilon(w) \exp\left(-\frac{2\pi i}{k+h} p \cdot w(p')\right). \tag{7.3.12}$$

Here Λ^w, Λ^r are the weight and root lattices of $\mathfrak{g}$, $\mathrm{Vol}\,\Lambda^i$ is the volume of a unit cell of the lattice, $|\Delta_+|$ is the number of positive roots and $\rho = \frac{1}{2}\sum_{\lambda_i>0}\lambda_i$ is the Weyl vector: it is half the sum of positive roots. For all the details on these notions, see [21]. In the first line,

$$h_p = \frac{p^2 - \rho^2}{2(k+h)} = \frac{c_2(R)}{(k+h)}. \tag{7.3.13}$$

is the conformal weight of the primary field associated to the highest weight state $|p\rangle$.* In the second line, the sum runs over elements w in the Weyl group $\mathcal{W}$, the subgroup of isometries generated by reflections in hyperplanes orthogonal to the roots of G. For example, one finds that this specializes for $G = SU(N)$† to

$$S_{mn} = \sqrt{\frac{2}{k+N}} \sin\left(\frac{(m+1)(n+1)\pi}{k+N}\right). \tag{7.3.15}$$

Computing the Jones polynomial

Now suppose we have links in S^3. For concreteness, we shall take $G = SU(N)$ and all punctures in the fundamental representation F from now on. We can split S^3 with a knot K within as follows: we can isolate a crossing in a knot projection, and draw a sphere S^2 around that crossing. This divides S^3 into two parts M_{out} and M_+, separated by an S^2 with four punctures $\{p_i\}_{i=1,\dots,4}$. Giving the knot K an orientation, two punctures will correspond to some fundamental representation F and the other two to the dual $\overline{F}$. By the axioms of topological field theory, we will get two vectors: χ associated to M_{out} and ψ_+ associated to M_+, from which we get

$$Z(S^3, K) = (\chi, \psi_+). \tag{7.3.16}$$

*We recall that for a representation $\mathbf{R}$ the quadratic Casimir number $c_2(\mathbf{R})$ is defined as

$$c_2(R) = d_{\mathbf{R}} \frac{\dim G}{\dim \mathbf{R}}, \qquad \mathrm{tr}\, t^a t^b = d_r \delta^{ab}. \tag{7.3.14}$$

This result follows by Schur's lemma: any object in the Lie algebra $\mathfrak{g}$ that commutes with all the generators, is necessarily proportional to the identity $\mathbb{I}$. We consider the operator $t^a_{\mathbf{R}} t^a_{\mathbf{R}}$. We have $[t^a, t^b t^b] = i f^{abc} t^c t^b + t^b i f^{abc} t^c = i f^{abc}\{t^b, t^c\} = 0$, since $f^{abc} = -f^{acb}$. Therefore $t^a_{\mathbf{R}} t^a_{\mathbf{R}} = c_2(\mathbf{R})\mathrm{id}$ and so $\mathrm{tr}\, t^a_{\mathbf{R}} t^a_{\mathbf{R}} = c_2(\mathbf{R}) \dim \mathbf{R}$. But we also have $\mathrm{tr}\, t^a t^a = d_r \delta^{aa} = d_r \dim G$. Comparing the two expressions gives the quoted result.

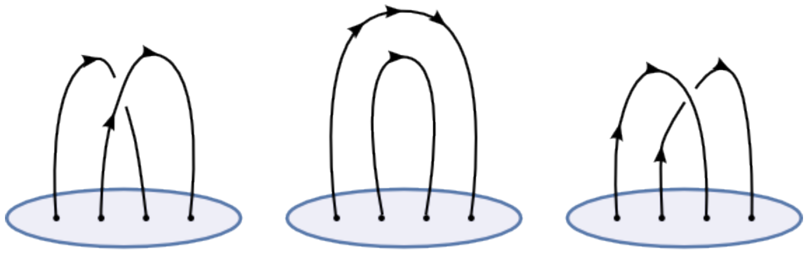
† For the case of $\mathfrak{su}(2)_k$, we have the data $r = 1, |\Delta_+| = 1, \left(\frac{\mathrm{Vol}\,\Lambda^w}{\mathrm{Vol}\,\Lambda^r}\right)^{\frac{1}{2}} = \sqrt{2}, h = 2$. The last two equations follow from the fact that $\mathfrak{su}(2)$ has only 2 roots lying in a unit cell of the weight lattice and that the dual Coxeter number of $SU(N)$ equals N.

The Hilbert space $\mathcal{H}_4$ associated to $S^2 \cup \{p_1, p_2, p_3, p_4\}$ with representations $F, F, \overline{F}, \overline{F}$ was 2. We now replace M_+ by M_0 or M_-, in which the punctures are connected by oriented lines such that locally they look like $L_0, L_\pm$ (see below). These will be assigned vectors $\psi_0, \psi_- \in \mathcal{H}_4$ by the topological field theory. But since we now have three vectors $\psi_+, \psi_0, \psi_- \in \mathcal{H}_4$, they must be linearly dependent, i.e. there are some constants c_+, c_0, c_- such that

$$c_+\psi_+ + c_0\psi_0 + c_-\psi_- = 0 \tag{7.3.17}$$

from which we infer, dropping the label S^3 from $Z(S^3, .)$,

$$c_+(\chi, \psi_+) + c_0(\chi, \psi_0) + c_-(\chi, \psi_-) = 0 \Longrightarrow c_+Z(M_+) + c_0Z(M_0) + c_-Z(M_-) = 0. \tag{7.3.18}$$



This equation can be massaged into the standard *skein relation* for the HOMFLY polynomial. The skein relation allows one to compute the HOMFLY polynomial recursively in the number of crossings. Note that this relation tells us that for the case of 2 unlinked unknots C, we have

$$c_+Z(C) + c_0Z(C^2) + c_-Z(C) = 0 \Longrightarrow \langle C\rangle = -\frac{c_+ + c_-}{c_0}. \tag{7.3.19}$$

One can determine the coefficients c_+, c_0, c_- by translating the crossing diagrams for M_+, M_0, M_- into braid operations in $\mathcal{H}_4$. After taking framing issues into account, one finds, after setting $q = \exp \frac{2\pi i}{N+k}$, that the skein relation can be written as

$$-q^{\frac{N}{2}}Z(M_+) + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})Z(M_0) + q^{\frac{N}{2}}Z(M_-) = 0. \tag{7.3.20}$$

The details of this calculation can be found in [19]. We can check that our values found for c_+, c_0, c_- correctly give for the unknot C *

$$\langle C\rangle = \frac{q^{\frac{N}{2}} - q^{-\frac{N}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}. \tag{7.3.22}$$

In knot theory, it is conventional to normalize $\langle C\rangle = 1$. For $G = SU(2)$, the knot invariant obtained in this way is called the *Jones polynomial* $J(q)$. It is a Laurent polynomial in q and has *integer coefficients*. We will come back to this in section 8. For $G = SU(N)$, the knot invariant is called the HOMFLY polynomial, while for $G = SO(N)$, it is called the Kauffman polynomial.‡

*One check of this formula: in the weak coupling limit $k \longrightarrow \infty, q \longrightarrow 1$, the correlation function should go to its value for $A = 0$, since the fluctuations in the connection shouldn't matter any more at weak coupling. But that value is just the dimension of the representation since in this case

$$\langle C\rangle = \langle \mathrm{tr}_R\, \mathrm{P}\exp \int A\rangle = \langle \mathrm{tr}_R\, 1\rangle = N \tag{7.3.21}$$

This answer is easily checked by employing l'Hopital's rule to equation (7.3.22).

‡The most important difference with the case $G = SU(N)$ is that the Hilbert space $\mathcal{H}$ associated to the sphere with four punctures is 3-dimensional, instead of 2. This means that we need a linear relation between 4 different vectors in $\mathcal{H}$: $a\psi_1 + b\psi_2 + c\psi_3 + d\psi_4 = 0$. These vectors are conveniently associated with the skein relation that defines the *Kauffman polynomial*.

Knot	$J(q)$
Trefoil	$-q^4+q^3+q$
Figure-eight	$q^2+\frac{1}{q^2}-q-\frac{1}{q}+1$
Solomon's seal or Cinquefoil	$-q^7+q^6-q^5+q^4+q^2$
Stevedore	$q^4-q^3+q^2+\frac{1}{q^2}-2q-\frac{1}{q}+2$

Table 4: The Jones polynomial $J(q)$ for some knots of low degree.

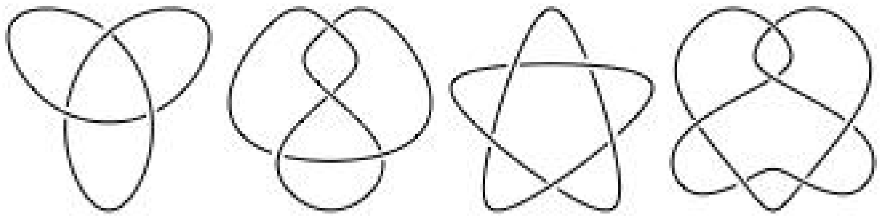


Figure 10: The trefoil knot, the figure-eight knot, Solomon's seal and the Stevedore knot.

Links on compact manifolds

For general compact manifolds, the following mathematical result without proof is key: any compact 3-manifold can be obtained from S^3 by a sequence of cutting and pasting. If we have a knot K in a general manifold M we can repeatedly excise a solid torus T_s from M, perform some diffeomorphism F on its boundary and glue it back into $M - T_s$, and obtain $\tilde{M} = (M - T) \cup_K T_s$, so that at some point, we will arrive at S^3. Suppose we perform one such operation and have a Wilson loop on a knot K in some representation R_i. A diffeomorphism F acting on $\partial T_s = T^2$ will lift to a CFT operator that acts on the representations R_i, so that the partition function transforms as

$$Z(\tilde{M}; R_i) = \sum_j F_i^{\ j} Z(M, R_j). \tag{7.3.23}$$

Note that this procedure is not unique: there are many ways to get from any closed M to S^3.

Links on non-compact manifolds

In the case of non-compact M, these techniques work analogously. Taking the special example of $M = \mathbb{R}^3$, we can compactify it by adding a point at infinity, which gives us $S^3 = \mathbb{R}^3 \cup \{\infty\}$. In doing field theory on $\mathbb{R}^3$, it is physical that one should consider only the connections that are trivial at infinity: $A|_\infty = 0$ and to gauge transformation that are 1 there. Hence, when we would use Chern-Simons theory to compute correlation functions on $\mathbb{R}^3$, we expect that it would give the same as Chern-Simons theory on S^3 with the constraint that we should only consider connections that are 0 at some specified point, the compactification point $\infty \in S^3$. But in doing Chern-Simons theory on S^3, the only connections that we consider are the (gauge equivalence classes of) flat connections, of which there is only 1 since the fundamental group $\pi_1(S^3) = 0$ is trivial, like on $\mathbb{R}^3$. Hence we infer that Chern-Simons theory on $\mathbb{R}^3$ should compute the exact same correlation functions or knots polynomial as Chern-Simons theory on S^3.

8

The exotic bulk-boundary duality

In this chapter we show that Chern-Simons theory is dual to twisted SYM, by generalizing and applying the techniques developed before. It is entertaining to see that in this way a Schwarz-type and a Witten-type topological theory are related. We explain what the geometric setting of the duality is: one can realize the duality between the two gauge theories in string theory. Using non-perturbative string dualities then allows us to describe a proposition for a gauge theory description of Khovanov homology.

8.1 Flow equations and critical orbits in Chern-Simons theory

In this section, we shall assume that the SYM θ-angle is 0. Recall that the physical fields in Chern-Simons theory are the gauge fields A, that are connections on a G-bundle $E \longrightarrow M$, where G is a compact Lie group and M is a 3-manifold. Our first goal is to find an alternative expression for the path integral (7.1.3) (the WRT-invariant) and knot correlators. We shall first analyze the situation in the case that there are no knots inserted on M:

$$Z_{CS}(M) = \int_{\mathcal{M}} \mathcal{D}A \exp\left(iS_{CS}(A)\right) = \int_{\mathcal{M}} \mathcal{D}A \exp\left(i\frac{k}{4\pi}\int_M \mathrm{tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right)\right) \quad (8.1.1)$$

where we made explicit that we integrate over the phase space

$$\mathcal{C}_{\mathbb{R}} = \mathcal{M} = \{\text{Gauge fields } A \text{ on } E \to M\},$$

which is infinite-dimensional. We call this normal integration cycle $\mathcal{C}_{\mathbb{R}}$. Note that adding Wilson loops does not affect the convergence of the path integral on the exotic cycle defined by (8.1.7) since a Wilson loop is linear in the gauge field $\mathcal{A}$ in the exponent. Hence, we will not include them for now; we will come back to them later. According to the recipe developed in the previous chapters, our first step is now to complexify everything. So we consider a $G_{\mathbb{C}}$-bundle, the complexification $E_{\mathbb{C}} \longrightarrow M$, on which we have the phase space

$$\mathcal{M}_{\mathbb{C}} = \{\text{Gauge fields } \mathcal{A} = A + i\phi \text{ on } E_{\mathbb{C}} \to M\},$$

where ϕ is an $\mathrm{ad}(E)$-valued 1-form. The curvature of $\mathcal{A}$ is $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ and the complexified action is

$$\mathcal{S}_{CS} = re^{-i\alpha}\int_M d^3x \left(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\right). \quad (8.1.2)$$

Since $G_{\mathbb{C}}$ is complex, the level k does not have to be an integer anymore in general, so we can write $\frac{ik}{4\pi} = re^{-i\alpha}$. To write down flow equations, we need a choice of metric g on M: a Kähler metric on $\mathcal{M}_{\mathbb{C}}$ is given by

$$ds^2 = -\int_M \mathrm{tr}\left(\delta\overline{\mathcal{A}} \wedge *_M \delta\mathcal{A}\right), \quad (8.1.3)$$

where $\overline{\mathcal{A}} = A - i\phi$ denotes the complex conjugate connection. Its Kähler form is

$$\omega = \int_M \mathrm{tr}\left(\delta\phi \wedge *_M \delta A\right). \quad (8.1.4)$$

An infinitesimal G gauge transformation is $\mathcal{A} \longmapsto \mathcal{A} + \delta\mathcal{A} = \mathcal{A} + d_A\lambda$ with λ a $\mathfrak{g}$-valued function.*

*A finite G gauge transformation maps $\mathcal{A} \to g\mathcal{A}g^{-1} + dgg^{-1}$. Expanding around the identity, we write $g = 1 + \epsilon\lambda + \ldots$. It follows that infinitesimally $\mathcal{A} \to \mathcal{A} + \epsilon[A, \lambda] + \mathcal{O}(\epsilon^2) = \mathcal{A} + d_A\lambda$, since G only acts on the real part of $\mathcal{A}$.

Hence, the vector field that generates a real complex transformation is $X = d_A\lambda\frac{\delta}{\delta A}$. It follows from a suitable generalization [26] of the formula $d\mu_G = \iota_X\omega$ that the moment map for G gauge transformations is defined by

$$\int_M \mathrm{tr}\,(\delta\mu_G\lambda) = \iota_Z\omega = -\int_M \mathrm{tr}\,(d_A\lambda \wedge *_M\delta\phi) = \int_M \mathrm{tr}\,(\delta(d_A *_M \phi)\lambda) \tag{8.1.5}$$

where we used that δ and d_A commute and that λ as a $\mathfrak{g}$-valued function can be freely moved around in the last equation. It follows that the moment map for the G-action is the 3-form $\mu_G = d_A *_M \phi$.*

So we want to take the real part of (8.1.2) as a Morse function $h = \mathrm{Re}\,\mathcal{S}_{CS}$ and look for its critical subsets. These subsets correspond to $G_\mathbb{C}$-orbits: gauge equivalence classes of flat $G_\mathbb{C}$-connections for which $\mathcal{F} = 0$. We already saw that the interesting critical orbits were the semistable ones: for a flat $G_\mathbb{C}$-connection $\mathcal{A}$ to be gauge equivalent to a connection for which $\mu_G = 0$, means that the holonomy of $\mathcal{A}$ around 1-cycles on M is not strictly triangular [27].

In terms of the complex gauge fields $\mathcal{A}, \overline{\mathcal{A}}$, using the metric (8.1.3) the flow equation now becomes

$$\frac{d\mathcal{A}}{ds} = -\exp(-i\alpha) *_M \frac{\delta\overline{\mathcal{S}}_{CS}}{\delta\overline{\mathcal{A}}} = -\exp(-i\alpha) *_M \overline{\mathcal{F}}. \tag{8.1.7}$$

We can think of the functional derivative as the coefficient that multiplies $\wedge\delta\mathcal{A}$ in varying $\mathcal{S}_{CS}$ with respect to $\mathcal{A}$. Note that we can always rescale s so that the level ik has norm 1, so we can set $r = 1$ in (8.1.2). It is now straightforward algebra to show that these equations are equivalent to

$$(F - \phi\wedge\phi)^+ = t(d_A\phi)^+, \qquad (F - \phi\wedge\phi)^- = -t^{-1}(d_A\phi)^-, \tag{8.1.8}$$

where $d_A = d + [A, .]$ is the gauge-covariant derivative, $\pm$ denote the self-dual and anti-self-dual projections and we defined the constants

$$t = \frac{1-\cos\alpha}{\sin\alpha}, \qquad t^{-1} = \frac{1+\cos\alpha}{\sin\alpha}. \tag{8.1.9}$$

The precise calculation can be found in appendix D.3.

Having obtained the flow equations, we would now like to find an exotic integration cycle for Chern-Simons theory as before by studying downward flow from critical orbits of $h = \mathrm{Re}\,\mathcal{S}_{CS}$. The procedure for this would be entirely analogous to the one outlined in chapter 6, however the phase space $\mathcal{M}_\mathbb{C}$ is infinite-dimensional. So at this point we again need to apply the Morse theory techniques in the infinite-dimensional case, do Floer theory. Recall from chapter 6 that the reason that we can apply Floer theory is that the flow equations should be elliptic differential equations. (8.1.7) is elliptic, since it involves the differential operator $\frac{1}{2}(1+*)d_A \equiv d_A^+$, whose symbol is simply a linear function p, that vanishes at the origin.

The space of physical states needed in that procedure is given by the equivariant cohomology of $\mathcal{M}$, where the group action now is given by local gauge transformations. Every stable critical orbit contributes a state to the equivariant cohomology $H^*_G(\mathcal{M})$: similar observations from in 6 hold analogously.

*The metric (8.1.3) is G-invariant, but not $G_\mathbb{C}$-invariant: under a gauge transformation

$$\mathcal{A} \longmapsto g\mathcal{A}g^{-1} + dgg^{-1}, \qquad \overline{\mathcal{A}} \longmapsto \overline{g}\overline{\mathcal{A}}\overline{g}^{-1} + d\overline{g}\overline{g}^{-1}. \tag{8.1.6}$$

the metric picks up a term $\int_M \mathrm{tr}\left(\overline{g}\delta\overline{\mathcal{A}}\overline{g}^{-1} \wedge *_M g\delta\mathcal{A}g^{-1}\right)$; since $\overline{g}g^{-1} \neq 1$, it is clear that in general the metric is not invariant under a complex gauge transformation. However, under a real gauge transformation, the metric is invariant, since in that case $\overline{g} = g$. We can spell this out more clearly in the abelian case: under $A \longmapsto A + d\lambda$ the metric then transforms as

$$-\int_M \mathrm{tr}\,(\delta\overline{\mathcal{A}} \wedge *_M\delta\mathcal{A}) - \int_M \mathrm{tr}\,(\delta d\lambda \wedge *_M\delta A + \delta A \wedge *_M\delta d\lambda) + \mathcal{O}((\delta\lambda)^2) = \ldots - \underbrace{\int_M \mathrm{tr}\,(\delta d\lambda \wedge *_M\delta A - *_M\delta d\lambda \wedge \delta A)}_{=0},$$

where the minus sign comes in since we regard δA and $\delta d\lambda$ as 1-forms. Therefore, we have at most a moment map G gauge transformations.

Exotic integration cycle

Semistability requires that the moment map for real G-gauge transformations vanishes on the critical orbit, so we find that in general the equations for a semistable critical orbit are

$$\frac{k\mathcal{F}}{2\pi} = 0, \qquad d_A *_M \phi = 0. \tag{8.1.10}$$

It is now straightforward to use the techniques from chapter 5 to find an exotic integration cycle for Chern-Simons theory. The original integration cycle was the space $\mathcal{C}_{\mathbb{R}}$ of all real gauge fields: $\mathcal{A} = A, \phi = 0$. This cycle is expressed in terms of Lefschetz thimbles $\mathcal{C}_\sigma$, associated to the critical orbits $\mathcal{O}_\sigma$, as

$$\mathcal{C}_{\mathbb{R}} = \sum_\sigma n_\sigma \mathcal{C}_{\mathcal{O}_\sigma}. \tag{8.1.11}$$

Since h is the real part of a complexification, it is a perfect Morse function, so there will be no interpolating flows between critical orbits. From this decomposition, we find the exotic integration cycle. In general, it is hard to find the coefficients n_σ. However, for manifolds M for which $\pi_1(M)$ is trivial, there is only one critical orbit: $A = \phi = 0$, up to gauge transformations. As explained above (4.2.14), $n_{\text{pure gauge}} = 1$: we see that the exotic cycle is equivalent to the original one, consisting of all real gauge fields. Gauge transformations act freely on pure gauge connections, therefore no subtle issues arise from G-fixed points. So $\mathcal{C}_{\mathbb{R}} = \mathcal{C}_{\text{pure gauge}}$, where equality again is at the level of path integrals.

8.2 Twisted $\mathcal{N} = 4$ SYM

Now the main feature of the path integral duality is that solutions to the Morse flow equations correspond to the class of maps that the dual topological theory localized on. Here we proceed to construct the TFT that localizes on the flow equations (8.1.8) for Chern-Simons theory on M. We shall also now allow θ to be non-zero. From chapter 5, we heuristically know what the dual theory should be: the open 1-dimensional gauged σ-model whose target space is the space of complexified connections $\mathcal{M}_{\mathbb{C}}$ on $E_{\mathbb{C}} \to M$. The gauged symmetry is just the space of gauge transformations on $\mathcal{M}$. Its superpotential should be the Chern-Simons action, which gives the right Morse function. Upon localization we get a residual boundary integration, which is exactly the Chern-Simons path integral over a middle-dimensional exotic integration cycle in the space of complex connections.

It turns out that this σ-model can be obtained from twisted $\mathcal{N} = 4$ SYM on $Z = \mathbb{R}_- \times M$ whose θ-angle vanishes. When $\theta \neq 0$, a slight modification is needed in the boundary conditions on ∂Z. We now proceed to show that the flow equations for Chern-Simons theory are exactly the localization equations for twisted 4d $\mathcal{N} = 4$ SYM. The reference for this material is [17].

We start out by compactifying $\mathcal{N} = 4$ SYM on $\mathbb{R}_- \times M$. If M is curved, we need to accompany this by a supersymmetric twist to preserve some supersymmetry. The 4-dimensional supersymmetries transform under Spin$(4) \times$ Spin$(6) \cong SL(2) \times SL(2) \times$ Spin(6) as $\mathbf{(2,1,\bar{4})} \oplus \mathbf{(1,2,4)}$ (see [17]). As always, we need to twist using the R-symmetry of the theory. The R-symmetry group of 4d $\mathcal{N} = 4$ SYM is $SU(4) \cong SO(6)$. Considering first $\mathcal{N} = 4$ SYM on flat $\mathbb{R}^4$, the Lorentz group is $SO(4)$, while the symmetry group of $\mathcal{N} = 4$ SYM is the larger Spin$(4) \times$ Spin$(6)_R$, where we have taken the double cover of $SO(6)$, as the fermions sit in Spin-representations. So twisting means that we need to choose a diagonal embedding Spin$(4) \times$ Spin$(4) \subset$ Spin$(4) \times$ Spin$(6)_R$ such that we again get Lorentz scalar fermions. This situation differs qualitatively from the A-model in that we now have 3 inequivalent choices of diagonal embeddings, whereas with the A-model we only had one (up to signs).

It turns out that the appropriate twist for our setup is the *geometric Langlands twist*, which is described in [17]. To define the twist, we need to specify a homomorphism h : Spin$(4) \to$ Spin$(6)_R = SU(4)_R$, upon which the new Lorentz group is given by Spin$(4)' = (1 \times h)$Spin(4). The idea is to choose h such that we embed Spin$(4) = SU(2)_l \times SU(2)_r$ as

$$\begin{pmatrix} SU(2)_l & 0 \\ 0 & SU(2)_r \end{pmatrix}, \tag{8.2.1}$$

which commutes with a $U(1)$ group whose generator is

$$F = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{8.2.2}$$

This residual $U(1)$ can be interpreted as the residual Spin(2) R-symmetry. Using this embedding the **4** of Spin(6) transforms under $\text{Spin}(4)' \times U(1)$ as $\mathbf{(2,1)}^1 \oplus \mathbf{(1,2)}^{-1}$ and the $\overline{4}$ as $\mathbf{(2,1)}^{-1} \oplus \mathbf{(1,2)}^1$. It follows that the 4-dimensional supersymmetries transform under the new $\text{Spin}(4)'$ as

$$(\mathbf{2},\mathbf{1},\overline{\mathbf{4}}) \longrightarrow (\mathbf{2},\mathbf{1})^0 \otimes \left((\mathbf{2},\mathbf{1})^{-1} \oplus (\mathbf{1},\mathbf{2})^1\right) \longrightarrow (\mathbf{1},\mathbf{1})^{-1} \oplus (\mathbf{3},\mathbf{1})^{-1} \oplus (\mathbf{2},\mathbf{2})^1 \tag{8.2.3}$$

$$(\mathbf{1},\mathbf{2},\mathbf{4}) \longrightarrow (\mathbf{1},\mathbf{2})^0 \otimes \left((\mathbf{2},\mathbf{1})^1 \oplus (\mathbf{1},\mathbf{2})^{-1}\right) \longrightarrow (\mathbf{1},\mathbf{1})^{-1} \oplus (\mathbf{1},\mathbf{3})^{-1} \oplus (\mathbf{2},\mathbf{2})^1. \tag{8.2.4}$$

We see in this decomposition that both representations contain a factor of **(1,1)**, hence there are two generators ϵ_l (coming from $\mathbf{(2,1,\overline{4})}$) and ϵ_r that are invariant under $\text{Spin}(4)'$. Using chirality relations and properties of 10-dimensional Γ-matrices, it turns out that we can pick any complex linear combination $\epsilon = u\epsilon_l + v\epsilon_r$ and declare this to be the topological supersymmetry parameters; since a rescaling of ϵ is irrelevant, we see that the topological symmetry is parametrized by $t = \frac{v}{u} \in \mathbb{C}P^1$. The associated supersymmetry generator is $Q = uQ_l + vQ_r$, and it is convenient to write $\delta = \epsilon_l \{Q_l, .\} + \epsilon_r \{Q_r, .\}$. For the fermionic fields one then finds the supersymmetry variations (we drop the indices for clarity)

$$\begin{aligned}
\delta\chi^+ &= u\,(F-\phi\wedge\phi)^+ + v(d_A\phi)^+, & \delta\eta &= vd_A^*\phi + u\,[\overline{\sigma},\sigma]\,, & \delta\psi &= ud_A\sigma + v\,[\phi,\sigma]\,,\\
\delta\chi^- &= v\,(F-\phi\wedge\phi)^- - u(d_A\phi)^-, & \delta\tilde{\eta} &= -ud_A^*\phi + v\,[\overline{\sigma},\sigma]\,, & \delta\tilde{\psi} &= vd_A\sigma - u\,[\phi,\sigma]\,.
\end{aligned}$$

Here $d_A = d + [A,.]$, $d_A^*\phi = *d_A * \phi = d_{A,\mu}\phi^\mu$ and $\sigma = \frac{1}{\sqrt{2}}(\phi_4 - i\phi_5)$. Note that χ is a 2-form, η a 0-form and ψ a 1-form.

From this we see what we claimed in the previous paragraph: the Q-fixed points of $\chi^\pm$ are equivalent to the flow equations (D.3.4) of Chern-Simons theory! Since Q is nilpotent, it follows from the discussion in chapter 3 that the path integral of the twisted SYM theory can be localized on field configurations that obey $\delta\Phi = 0$. This immediately shows that the philosophy outlined in chapter 6 holds: the exotic dual theory localizes on solutions to the flow equation of the theory one starts out with. For convenience we define

$$\mathcal{U}^+ = (F-\phi\wedge\phi+td_A\phi)^+\,, \qquad \mathcal{U}^- = \left(F-\phi\wedge\phi-t^{-1}d_A\phi\right)^-\,, \qquad \mathcal{U}^0 = d_A^*\phi, \tag{8.2.5}$$

so the theory localizes on $\mathcal{U}^+ = \mathcal{U}^- = \mathcal{U}^0 = 0$. It is found in [17] that the topological SYM action can be constructed by brute force as

$$S_{SYM}^{tw} = \{Q, Z\} + i\mathbf{k}\int_{\mathbb{R}_-\times M} \text{tr}\,(\mathcal{F}_w \wedge \mathcal{F}_w) \tag{8.2.6}$$

where $\mathcal{F}_w$ is the curvature of the complex connection $\mathcal{A}_w = A + w\phi, w \in \mathbb{C}$ and the *canonical parameter* $\mathbf{k}$ is defined as

$$\mathbf{k} = 4\pi i\left(\frac{1}{g_{10}^2}\frac{t-t^{-1}}{t+t^{-1}} - \frac{i\theta}{8\pi^2}\right). \tag{8.2.7}$$

The canonical parameter is related to the Yang-Mills coupling constant (2.1.7), given by $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{\lambda^2}$ as

$$\mathbf{k} = \frac{\tau+\overline{\tau}}{2} + \frac{\tau-\overline{\tau}}{2}\left(\frac{t-t^{-1}}{t+t^{-1}}\right). \tag{8.2.8}$$

This action contains by construction the term

$$\frac{1}{\epsilon}\left\{Q, \int \text{tr}\left(\chi^+\mathcal{U}^+ + \chi^-\mathcal{U}^- + \chi^0\mathcal{U}^0\right)\right\} \subset S_{SYM}^{tw} \tag{8.2.9}$$

which ensures that the theory localizes as $\epsilon \to 0$ on the field configurations for which the equations in (8.2.5) vanish. Here, the χs are Q-exact fermions and we dropped indices for clarity. The lengthy details can be found in [17, 29].

Using Stokes theorem and our calculations in chapter 7 we have $\int_{\mathbb{R}_-\times} \text{tr}\,(\mathcal{F}_w \wedge \mathcal{F}_w) = \int_M \text{CS}(\mathcal{A}_w)$. We therefore find that we can write the twisted action as

$$S^{tw}_{SYM} = \{Q, Z\} + i\mathsf{k}\int_M \text{CS}(\mathcal{A}_w) \equiv -\mathcal{S}^{top}_{SYM} + i\mathbf{k}\mathcal{I}_{CS}(\mathcal{A}_w), \qquad \mathcal{I}_{CS}(\mathcal{A}_w) = \int_M \text{CS}(\mathcal{A}_w). \quad (8.2.10)$$

8.3 Branes and boundary conditions

To motivate the procedure of the final parts of this chapter, we will first describe how to embed the 'Chern-Simons+SYM'-system for $G = U(N)$ into type IIB superstring theory, using a NS5-D3 brane system. For some background on these objects, see appendix ?? and the text [2].

So we put type IIB superstring theory on $\mathbb{R}^2 \times T^*Z'$, where Z' is a 4-manifold. We assume that T^*Z' is Calabi-Yau. Inside this space we have $\mathbb{R}_- \times Z' \subset \mathbb{R}^2 \times T^*Z'$, where Z' is embedded as the zero section. We then wrap N D3-branes on $Z' \times \{0\}$, which is described by $\mathcal{N} = 4$ SYM, suitably twisted to preserve supersymmetry on Z. We make a choice of 3-manifold M such that $T^*M \subset T^*Z'$ is a complex submanifold. M might or might not divide Z' in two pieces, such that $M = \partial Z$ for some Z. The special choice in the previous section was $Z = \mathbb{R}_- \times M$. Since the 6-manifold T^*M is a Lagrangian submanifold, we can wrap an NS5-brane (see appendix ??) on T^*M, on which the D3's end. This last condition is sufficient for the NS5 brane to preserve maximal supersymmetry, as shown by a non-trivial calculation in [30].

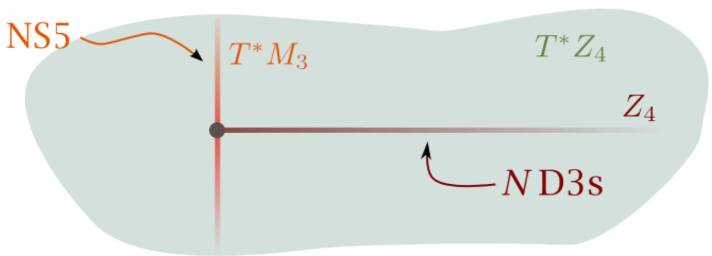


Figure 11: Geometry of the NS5-D3 brane system.

Supersymmetric boundary conditions when $\theta \neq 0$

We described the brane system that is the string picture of the duality between Chern-Simons and $\mathcal{N} = 4$ SYM. Here, the D3-branes were supported on Z. For our purposes, we need to allow for a non-zero θ-angle. So what is the $\frac{1}{2}$-BPS boundary condition on the SYM bosons and fermions at ∂Z when $\theta \neq 0$, such that maximal supersymmetry* is preserved at ∂Z? We shall see that the appropriate boundary partially fixes the behavior of the fields at ∂Z.

We recall from equation (2.2.6) that in $\mathcal{N} = 4$ SYM, using Noether's theorem, the supercurrent associated to supersymmetry is

$$J^I = \frac{1}{2}\,\text{tr}\left(\Gamma^{JK} F_{JK} \Gamma^I \lambda\right), \quad (8.3.1)$$

*Note that on general grounds these should be $\frac{1}{2}$ BPS boundary conditions: the boundary breaks one translational symmetry, such that half of the supersymmetry is broken.

where λ was a 10-dimensional Weyl spinor and $I = 0, \ldots 9$. We recall that dimensional reduction $SO(1,9) \longrightarrow SO(1,3) \times SO(6)_R$ gives us

$$\text{a gauge field } A_i, i = 0, \ldots, 3, \qquad \text{6 scalars } A_{3+a} = \phi_a, a = 1,2,3; A_{6+m} = \zeta_m, m = 1,2,3 \tag{8.3.2}$$

and four Weyl spinors; λ transforms as **(2,1,4)** $\oplus$ **(1,2,$\bar{4}$)** under the broken gauge group. Furthermore, it will be useful to note that we can decompose the **16** spinor of $SO(1,9)$ as $Z_8 \otimes Z_2$, where Z_8 transforms as **(2,2,2)** of $SO(1,2) \times SO(3)_\phi \times SO(3)_\zeta$ and Z_2 is a 2-dimensional vector space. A boundary preserves supersymmetry iff it ensures that the component of the supercurrent normal to the boundary vanishes. Given a supersymmetry generator ϵ, the condition for the special choice $\mathbb{R}_- \times M$ for the uncompactified space is

$$\mathrm{tr}\left(\bar{\epsilon}\Gamma^{IJ}F_{IJ}\Gamma_s\lambda\right) = 0, \tag{8.3.3}$$

where the subscript s stands for the coordinate along $\mathbb{R}_-$, whose boundary is at $s = 0$. In general, this equation can only be satisfied at the boundary ∂Z for vectors $\epsilon \in Z_8 \otimes Z_2$. After compactification, the above restriction implies the following set of equations:

$$0 = \bar{\epsilon}\left(\Gamma^{ij}F_{ij} + 2\Gamma^{3i}F_{3i}\right)\tilde{\lambda}, \qquad 0 = \bar{\epsilon}\left(\Gamma^{im}D_i\zeta_m\right)\tilde{\lambda}, \qquad 0 = \bar{\epsilon}\left(2\Gamma^{3a}D_3\phi_a + \Gamma^{ab}\left[\phi_a, \phi_b\right]\right)\tilde{\lambda},$$
$$0 = \bar{\epsilon}\left(\Gamma^{ia}D_i\phi_a\right)\tilde{\lambda}, \qquad 0 = \bar{\epsilon}\Gamma^{am}\left[\phi_a, \zeta_m\right]\tilde{\lambda}, \qquad 0 = \bar{\epsilon}\left(2\Gamma^{3m}D_3\zeta_m + \Gamma^{mn}\left[\zeta_m, \zeta_n\right]\right)\tilde{\lambda}.$$

Here $\tilde{\lambda}$ is an 8-dimensional vector, the indices have ranges $i, j = 0,1,2$, $a, b = 1,2,3$, $m, n = 1,2,3$, and there is no summation over a, m. After exploiting consistency conditions and symmetries, as done in detail in [31], one finds that to satisfy the first condition at $s = 0$, the 4-dimensional gauge fields and fermions must satisfy

$$\epsilon_{ijk}F^{3k} + \gamma F_{ij} = 0, \qquad \bar{\epsilon}_0\left(1 + \gamma B_0\right)\theta = 0. \tag{8.3.4}$$

where γ is some constant and $B_0 = \Gamma_{456789} = \Gamma_4\Gamma_5\Gamma_6\Gamma_7\Gamma_8\Gamma_9$ and ϵ_0, θ are two 2-dimensional vectors*. If $\gamma = -\infty$, we get the constraint $F_{ij}|_{y=0} = 0$, which imply Dirichlet boundary conditions for A^i, while $\gamma = 0$ implies $\epsilon_{ijk}F^{3k} = 0$, which correspond to Neumann boundary conditions on A^k. It turns out that to satisfy all other conditions in a non-trivial way, exactly one of the bosonic fields ϕ, ζ has to satisfy Dirichlet boundary conditions, say ζ. Then we have

$$D_3\phi_a + \frac{u}{2}\epsilon_{abc}\left[\phi_b, \phi_c\right] = 0, \qquad 0 = \bar{\epsilon}_0\left(1 + uB_1\right)\theta, \qquad B_1 = \Gamma_{3456}, \tag{8.3.5}$$

for the same constant u as in the previous section (this will follow below). Since we can set the scale of ϵ arbitrarily, we can write $\bar{\epsilon}_0 = \begin{pmatrix}1 & a\end{pmatrix}$ for some parameter a, upon which the first constraint tells us that $\theta^t = \begin{pmatrix}1 & a\end{pmatrix}$. Writing out the constraints (8.3.4) and (8.3.5) then determines the constants γ and u to be

$$\gamma = -\frac{2a}{1 - a^2}, \qquad u = -\frac{2a}{1 + a^2}. \tag{8.3.6}$$

One can now interpret this as follows: (ϕ^a, A_i) are part of a vector multiplet, while (ζ^m, A_3) are part of a hypermultiplet. Let us now consider all possible values of a.

For $G = U(N)$, setting $a = 0, \infty$ means that $\gamma = u = 0$. Recall that this implies that the bosonic part (ϕ, A) of the vector multiplet obeys Neumann boundary conditions. Also, $\gamma = 0$ implies that A_3 vanishes, and we already assumed that ζ obeyed Dirichlet boundary conditions, so it vanishes too at the NS5-brane. All together, we get exactly the boundary conditions that arise for D3-branes ending on an NS5-brane located at $x^3 = x^7 = x^8 = x^9 = 0$ with vanishing gauge theory θ-angle. This was shown in [32].

*In this 2-dimensional vector space, $B_{0,1,2}$ are represented by $B_0 = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, B_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, B_2 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$.

If $a \neq \pm 1$, the boundary condition (8.3.4) arises from the bosonic part of the Yang-Mills action with a θ-term or instanton term added:

$$S_{YM} = \frac{1}{g^2}\int_{s\geq 0} d^4x \,\mathrm{tr}\left(\frac{1}{2}|F|^2\right) + \frac{\theta}{8\pi^2}\int_{s\geq 0}\mathrm{tr}\,(F\wedge F)\,. \tag{8.3.7}$$

Varying this action with respect to A and no restriction on A at $s=0$, we find that

$$\delta S_{YM} = \frac{1}{g^2}\int_{s\geq 0} d^4x\,\mathrm{tr}\left(\frac{1}{2}\delta(F_{ij}F^{ij})\right) + \frac{\theta}{8\pi^2}\int_{s\geq 0}\mathrm{tr}\left(\epsilon^{ijkl}\delta(F_{ij}F_{kl})\right) \tag{8.3.8}$$

$$= \frac{1}{g^2}\int_{s\geq 0} d^4x\,\mathrm{tr}\left(F_{ij}\delta F^{ij}\right) + 2\frac{\theta}{8\pi^2}\int_{s\geq 0}\mathrm{tr}\left(\epsilon^{ijkl}F_{ij}\delta F_{kl}\right) = 0. \tag{8.3.9}$$

so at $s=0$ we have that

$$\frac{1}{g^2}F_{ij} + 2\frac{\theta}{8\pi^2}\epsilon_{ijk3}F^{k3} = 0 \Rightarrow \gamma = -\frac{2a}{1-a^2} = -\frac{\theta g^2}{4\pi^2}. \tag{8.3.10}$$

We see that at $a \neq \pm 1$ the supersymmetric boundary condition $s=0$ implies that we need to add the topological θ-term to the worldvolume gauge theory for the D3-branes. Since the tangential part of the gauge field is a multiple of the normal part, we can view the case of generic a as a generalization of the Neumann boundary condition and the NS5-D3 bound state.

The last case is when $a = \pm 1$. In that case, the roles are reversed and (ζ, A_μ) are part of a vector multiplet satisfying Dirichlet boundary conditions, while (ϕ, A_3) are a vector multiplet. it turns out that this describes a system of D3s ending on multiple D5s.

Finally, after the geometric Langlands twist we only affect the fermions in the theory, therefore for the twisted theory the boundary conditions for the bosons are unaffected. Moreover, the boundary conditions are local, so they are not changed when M is curved. We already saw that the topologically twisted supersymmetry generator $\epsilon = \epsilon_l + t\epsilon_r$ had a free parameter t. ϵ satisfies

$$\left(1 + i\frac{1-t^2}{1+t^2}B_0 + \frac{2t}{1+t^2}B_1\right)\epsilon = 0. \tag{8.3.11}$$

It is straightforward algebra to check that the vector $\epsilon_0 = \begin{pmatrix}-a\\1\end{pmatrix}$ satisfies this relation too, if

$$a = i\frac{1-it}{1+it}. \tag{8.3.12}$$

Using this identification, it follows that using $\epsilon = \eta\otimes\epsilon_0$ and the representations of B_0, B_1 in Z_2, where η is a generic 8-dimensional vector, is a generator of the topological supersymmetry, which satisfies the $\frac{1}{2}$-BPS boundary condition of the NS5-D3 system with $\theta \neq 0$. Also, inserting (8.3.12) into (8.3.10) it is straightforward to check that

$$t^2 = \frac{\overline{\tau}}{\tau}. \tag{8.3.13}$$

Hence, we see that a choice of θ immediately specifies a and t.

8.4 Duality

We now put all the ingredients of the previous sections together. A priori, it is clear that the dual theory to Chern-Simons theory on M is the 1-dimensional gauged σ-model with target space $\mathcal{M}_{\mathbb{C}} = \{\text{Connections on } E_{\mathbb{C}} \to M\}$ and symmetry group Maps$(M, G_{\mathbb{C}})$. As one should expect, the superpotential W in this gauged σ-model is exactly the Chern-Simons action. From (5.2.6) we find that the associated Morse function is $h = A_1\mu + 2\mathrm{Re}\,W$, where the first term is inessential since on semistable orbits,

$\mu = 0$ and μ is preserved along downward flows (see (5.1.5)). Hence, effectively $h = 2\text{Re}\, W$, precisely as was suggested in section 8.1. Hence this model localizes on the flow equation for Chern-Simons theory.

Analogous to the finite setting in section 4.4, the 1-dimensional point of view is that the twisted $\mathcal{N} = 4$ SYM path integral on the half-space $\mathbb{R}_-$ computes a Poincaré dual to the exotic integration cycle $\Gamma_{\mathcal{O}}$ for Chern-Simons theory, which is set by the boundary condition at $s = -\infty$, where the gauge fields have to sit in a given critical orbit $\mathcal{O}$. The full exotic integration cycle then follows by appropriately summing the contributions coming from all Lefschetz thimbles $\mathcal{C}_{\mathcal{O}_\sigma}$.

The observation is that this 1-dimensional gauged σ-model can be constructed from compactification of $\mathcal{N} = 4$ SYM from $\mathbb{R}_- \times M$ on M with a partial supersymmetric twist on M, which leaves 4 scalar real supercharges. Hence, the dual 1-dimensional gauged σ-model is just the generalization to infinite-dimensional target space of the model discussed in section 5.2.1.

Recall from section 2.2.1 that the bosonic field content of 4-dimensional $\mathcal{N} = 4$ SYM is one gauge field $A_\mu, \mu = 0, \ldots 3$ and 6 scalars $\phi_i, i = 4, \ldots 9$. To go to the 1-dimensional point of view, we compactify on M with a partial topological twist. Let us discuss the bosonic fields. The twist is performed by embedding the $SO(3)$ Lorentz group of M in the $SO(6)$ R-symmetry group such that $\phi_i, i = 4, 5, 6$ become an adjoint 1-form (since they have non-trivial R-charge) and $\phi_j, j = 7, 8, 9$ remain scalars. After the partial twist, the residual components of the 4-dimensional gauge field A_μ are $A_a, a = 1, 2, 3$, which combine with the 1-forms ϕ_i into a complex connection $\mathcal{A}_\mu = A_\mu + i\phi_\mu$ on the bundle $E_{\mathbb{C}} \to M$. $\mathcal{A}$ acts as a coordinate on $\mathcal{M}_{\mathbb{C}}$ and therefore sits in a chiral multiplet. The gauge transformations on $E_{\mathbb{C}} \to M$ are gauged by the scalar fields A_0 and ϕ_j, which sit in a vector multiplet. Note that we can always go to a gauge where $A_0 = 0$. The fermions are redistributed likewise.

The complexified Chern-Simons action (8.1.2) arises naturally in this model as a superpotential: recall that the superpotential sits in the scalar potential Z as $|\delta W|^2 \subset Z$. Regarding W as a functional of the complex gauge field $\mathcal{A}(x)$, it follows from the calculations in appendix D that $\delta W = \mathcal{F}_{\mathcal{A}} \delta \mathcal{A}$. Using the metric (8.1.3), it follows that

$$|dW|^2 = r^2 \int_{\mathbb{R}_- \times M} \operatorname{tr} \mathcal{F}_{\mathcal{A}} \wedge \mathcal{F}_{\mathcal{A}} \tag{8.4.1}$$

gives exactly the kinetic term for the bosons in the chiral multiplet, from the 1-dimensional point of view (from the 10-dimensional point of view, this term was a kinetic term for the vector multiplet).

The 4-dimensional view of this duality is that supersymmetric localization means that the twisted $\mathcal{N} = 4$ SYM path integral on $\mathbb{R}_- \times M$ with action (8.2.10), analogous to the construction with the A-model, leaves us with a 'boundary integration': an integration over a middle-dimensional cycle in the space of complexified gauge fields of $E_{\mathbb{C}} \to M$, which is just the Chern-Simons path integral over an exotic integration cycle. The reason for middle-dimensionality is simply again that the Chern Simons flow equation uses the real part of the complexified Chern-Simons action as a Morse function. Downward flow then gives a middle-dimensional cycle, since the 'Morse index' of a critical orbit is exactly '$\frac{1}{2} \dim_{\mathbb{R}} \mathcal{M}_{\mathbb{C}}$'. The quotation marks indicate that really we should use the appropriate infinite-dimensional analogues given by Floer theory: here we emphasize the conceptual idea. At the boundary then, the set $\mathfrak{B}$ of boundary values of all solutions to the flow equation are middle-dimensional in the set of all possible boundary values. As familiar by now, $\mathfrak{B}$ is interpreted as an exotic integration cycle for the boundary theory.*

*Another example of this phenomenon is the following elliptic problem: given a holomorphic function on $\partial D = S^1$, by Cauchy's theorem it only extends to a holomorphic function on D if half of its Fourier-coefficients vanish, so only a middle-dimensional subspace of all holomorphic functions on S^1 extend to D holomorphically. This is an elliptic boundary value problem, since the Cauchy-Riemann equations are elliptic.

Without knots

After setting up all the ingredients in the previous sections, we can conclude that after localization we find the duality of path integrals

$$Z(\mathcal{O}) = \underbrace{\int_{\mathcal{C}_{\mathcal{O}}\subset\mathcal{M}_{\mathbb{C}}} \mathcal{D}\mathcal{A}\exp i\mathbf{k}\mathcal{S}_{CS}(\mathcal{A})}_{\text{Chern-Simons path integral}} = \underbrace{\int \mathcal{D}A\mathcal{D}\Phi\mathcal{D}\lambda \exp\left(-\mathcal{S}_{SYM}^{top}\right) \exp i\mathbf{k}\mathcal{S}_{CS}(\mathcal{A})|_{s=0}}_{\mathcal{N}=4 \text{ SYM path integral}}. \quad (8.4.2)$$

The boundary conditions are again left implicit in the notation. The full Chern-Simons partition function is then given by

$$Z(\mathcal{C}_{\mathbb{R}}) = \sum_{\sigma} n_{\sigma} Z(\mathcal{O}_{\sigma}), \quad (8.4.3)$$

where the expansion coefficients n_σ are as in (8.1.11). Now we need to consider the boundary conditions at $s = -\infty$. These are determined by the semistable critical orbits: flat complex connection with $\mu = 0$.[†]

The boundary conditions at $s = 0$ on the fermionic fields in the SYM theory on $\{s = 0\} \times M$ must be elliptic: the proof of this is in the appendix of [12]. The correct boundary conditions on the bosonic fields at $s = 0$ are those of the D3-NS5 brane system with non-zero θ-angle, as discussed in section 8.3 in (8.3.4) and (8.3.5). Note that these boundary conditions do not uniquely fix the values of the gauge fields at $s = 0$.

Now a straightforward example is given by taking $M = \mathbb{R}^3$ with no knots inserted. Note that in this case, the topological twist is not necessary to preserve supersymmetry, but is required to get a theory that localizes on the Chern-Simons flow equation. Recall from section 5.3.1 that semistable critical orbits may have flat directions that cause infrared divergences in perturbation theory. However, on $\mathbb{R}^3$ there is only one critical orbit: the equivalence class of the trivial connection consisting of pure gauge connections $A = dgg^{-1}$. This is a stable critical orbit since the equation $dgg^{-1} = 0$ only has $g =$ constant as a solution. The constant is fixed by requiring g to be 1 at infinity, which implies that indeed the stabilizer of G contains only the trivial group element, hence the critical cycle is stable. Phrased differently, a flat connection is specified by a homomorphism $v : \pi_1(M) \to G_{\mathbb{C}}$, which measures the monodromy of the flat connection around non-trivial cycles on M. Since $M = \mathbb{R}^3$ is simply-connected, v is trivial. So the boundary condition at $s = -\infty$ is just that the gauge fields approach $A = \phi = 0$. We see that by choosing $M = \mathbb{R}^3$ we can avoid semi-stable critical orbits and infrared divergences.

Recall that the Morse function h for Chern-Simons theory is perfect, as it is the real part of a complexification of the real Chern-Simons Lagrangian. Therefore, there are no interpolating flows and so the exotic integration cycle for CS without knots on $\mathbb{R}^3$ is equivalent to the original integration cycle, that is, we have $\mathcal{C}_{\mathbb{R}} = \mathcal{C}_{\text{pure gauge}} = \mathcal{C}$.

With knots

Now we want to generalize by adding knots. First we need to determine where we can insert Wilson loops on V. Consider twisted SYM on a general curved Z with twisting parameter t, with a non-zero θ-angle. A supersymmetric Wilson loop is a $\frac{1}{16}$-BPS Wilson loop operator

$$W_{K,R}(\mathcal{A}) = \operatorname{tr} \mathrm{P} \exp \oint_K (A + \xi\phi), \qquad \xi \in \mathbb{C}. \quad (8.4.4)$$

Note that we chose $\xi = i$ in (8.1), where we assumed that $\theta = 0$. Here, and ϕ is the 1-form obtained after twisting 4 of the 6 scalars of the $\mathcal{N} = 2$ theory. Specifically, one has $\phi_\mu = A_{4+\mu}$. One can show that at

[†] Note that we might have zero modes of the Dirac operator on Z, which forces us to insert extra fermions to absorb the zero modes. This would lead to an extra integration over the boundary, as the fermions represent differential forms constrained to the boundary. Hence a non-zero number of fermion zero modes make the exotic integration cycle at the boundary above middle-dimensional. We shall assume that no such zero modes exists, which will be true for our specialization to $Z = \mathbb{R}_- \times \mathbb{R}^3$ later on.

twisting parameter $t \neq \pm i$, such operators can only be inserted at ∂Z without breaking supersymmetry completely, due to the product nature of the fermions at ∂Z. Since (8.1.8) are only elliptic for real t, we see that we should insert knots K only at ∂Z.

For the Wilson loop (8.4.4) to be a good observable, it needs to be Q-closed for the topological supercharge Q:

$$[Q, W_{K,R}(\mathcal{A})] = 0. \tag{8.4.5}$$

The supersymmetry transformation for the gauge field is $\delta A_I = i\bar{\epsilon}\Gamma_I\lambda$. It turns out that if the twisting parameter $t = \pm i$, one can choose ϵ such that

$$(\Gamma_\mu + i\Gamma_{4+\mu})\,\epsilon = 0, \tag{8.4.6}$$

this was shown in [17]. So for $t = \pm i$, $W_{K,R}(\mathcal{A})$ is always a good supersymmetric observable. When $t \neq \pm i$, it is not. However, when $W_{K,R}(\mathcal{A})$ is supported on ∂Z, the fermions ϵ, λ can be written as a tensor product in $Z_8 \otimes Z_2$ as noted in section 8.3. Therefore, (8.4.6) can be reduced to

$$\theta^t\,(1 + i\xi B_0 B_1)\,\epsilon_0 = 0, \tag{8.4.7}$$

where the 2-dimensional vectors $\theta, \epsilon_0 \in Z_2$ are as in (8.3.5). By straightforward algebra one finds that the condition is satisfied when

$$\xi = i\frac{a^2 - 1}{a^2 + 1} = \frac{t - t^{-1}}{2} = \mp i\frac{\operatorname{Im}\tau}{|\tau|}. \tag{8.4.8}$$

Here t should be chosen to be real, to make (8.1.8) elliptic. Note that if $\theta = 0$, then $a = 0, \infty$ and so $\xi = i$, as expected. With this caveat, placing supersymmetric Wilson loops in ∂Z we get the equivalence of path integrals

$$\begin{aligned} Z(\mathcal{O}, \{K_i\}) &= \underbrace{\int_{\mathcal{C}_\mathcal{O} \subset \mathcal{M}_\mathbb{C}} \mathcal{D}\mathcal{A} \left(\exp i\mathbf{k}\mathcal{I}_{CS}(\mathcal{A}) \prod_i W_{K_i,R_i}(\mathcal{A}) \right)}_{\text{Chern-Simons path integral}} \\ &= \underbrace{\int \mathcal{D}A\mathcal{D}\Phi\mathcal{D}\lambda \exp\left(-\mathcal{S}^{top}_{SYM}\right) \left(\exp i\mathbf{k}\mathcal{I}_{CS}(\mathcal{A}) \prod_i W_{K_i,R_i}(\mathcal{A}) \right)\bigg|_{s=0}}_{\mathcal{N}=4\text{ SYM path integral}} . \end{aligned} \tag{8.4.9}$$

The boundary conditions are again left implicit in the notation. The full Chern-Simons correlator is then given by

$$\langle \prod_i W_{K_i,R_i}(\mathcal{A}) \rangle = Z(\mathcal{C}_\mathbb{R}, \{K_i\}) = \sum_\sigma n_\sigma Z(\mathcal{O}_\sigma, \{K_i\}), \tag{8.4.10}$$

where the expansion coefficients n_σ are as in (8.1.11). Now we need to consider the boundary conditions at $s = -\infty$. These are determined by the semistable critical orbits: flat complex connection with $\mu = 0$.

fermionicFor the bosonic fields, the elliptic boundary condition at $s = 0$ far away from the knot should be that of the NS5-D3 brane system. At the knot, they should approach a singular solution, that gives the right monodromy around the knot. At $s = -\infty$, the boundary condition is that the gauge fields approach the critical orbit $\mathcal{O}_\sigma$. Again, the most convenient choice now is to take $M = \mathbb{R}^3$, since $\pi_1(M)$ is trivial. There is only 1 critical orbit, the exotic integration cycle is equivalent to $\mathcal{C}_\mathbb{R}$ and all other observations made in the previous section hold.

9

Gauge theory and Khovanov homology

For $G = SU(2)$ Chern-Simons theory computes the Jones polynomial, which turns out to be a Laurent polynomial with integer coefficients. As it stands, Chern-Simons theory does not explain *why its coefficients are integers*. To explain this fact we describe a more powerful knot invariant than the Jones polynomial: its *categorification* known as *Khovanov homology*. Essentially, categorification means that we associate *a bi-graded vector space* to a knot instead of a number, in which taking a suitable trace gives back the Jones polynomial. In fancier words: we construct a chain complex whose Euler characteristic is a function of two variables t, q. At $t = -1$, the Euler characteristic is exactly the Jones polynomial. Therefore, the coefficients of the Jones polynomial are integers: they *count dimensions of vector spaces*. The reason that Khovanov homology is more powerful is that its chain complex is *bigraded*,

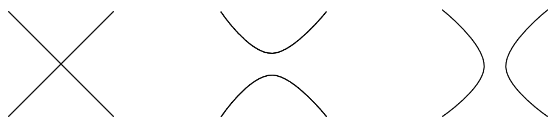
9.1 Khovanov homology: the construction

We shall be very pedestrian in our discussion of Khovanov homology, which is necessitated by the intricate mathematical framework needed to fully discuss the construction. Here we shall follow [22, 23, 24, 25] and work exclusively with links in $\mathbb{R}^3$.

First we describe the Jones polynomial in a different way. Earlier we defined it using the skein relation (7.3.20), here we give an alternative definition. Given a link projection L, we define $n_\pm$ as the number of $\pm$-crossings (as in the previous section) and let X be the set of all crossings in L. We set $n = n_+ + n_-$. The Jones polynomial may be defined through the Kauffman bracket $\langle . \rangle$ that satisfies

$$\langle \varnothing \rangle = 1, \qquad \langle \bigcirc L \rangle = \left(q^{\frac{1}{2}} + q^{-\frac{1}{2}} \right) \langle L \rangle, \qquad \langle \times \rangle = \langle 0 \rangle - q^{\frac{1}{2}} \langle 1 \rangle. \tag{9.1.1}$$

Here $\bigcirc$ is an unknot and, taking an unoriented crossing $\times$, we defined a 0-smoothing and a 1-smoothing as follows:



(a) $\times$ (b) 0-smoothing (c) 1-smoothing

For notational convenience, we now set $u \equiv q^{\frac{1}{2}}$. Moreover, we define:

$$\text{the unnormalized Jones polynomial: } \hat{J}_L(u) = (-1)^{n_-} u^{n_+ - 2n_-} \langle L \rangle,$$

$$\text{the normalized Jones polynomial: } J_L(u) = \frac{\hat{J}_L(u)}{u + u^{-1}}.$$

We now present the alternative: in the link projection, we forget the orientation and replace every link-crossing by either a 0-smoothing or a 1-smoothing. We get 2^n different smoothings $S_\alpha, \alpha \in \{0,1\}^X$ that consist of collections of topological unknots. We call the number of such unknots k and let the 'height' r be the number of 1-smoothings used in S_α. To every S_α we then assign the term $(-1)^r u^r (u + u^{-1})^k$. The Jones polynomial then follows as

$$J_L(u) = (-1)^{n_-} u^{n_+ - 2n_-} \cdot \frac{1}{u + u^{-1}} \cdot \sum_{\alpha \in X} (-1)^r u^r (u + u^{-1})^k. \tag{9.1.2}$$

The trefoil revisited.

The trefoil knot has $n_+ = 3, n_- = 0, n = 0$. There are 2^3 possible smoothings and their contributions are:

000	$(u+u^{-1})$	001	$3\cdot(-1)^1u(u+u^{-1})$
011	$3\cdot(-1)^2u^2(u+u^{-1})$	111	$(-1)^3u^3(u+u^{-1})$

Note that the factor of 3 come from the multiplicities. So we find that

$$\begin{aligned} &J_{\text{trefoil}}(u) \\ &= \frac{1}{u+u^{-1}}\left((-1)^0u^{3-2\cdot 0}\left((u+u^{-1}) - 3u(u+u^{-1}) + 3u^2(u+u^{-1})\right) - u^3(u+u^{-1})\right) \\ &= u^2 + u^6 - u^8. \end{aligned}$$

This can be compared to table 4, noting that $u = q^{1/2}$.

To get a homological complex, instead of assigning polynomials to a given smoothing, we want to assign a graded vector space to it. Graded vector spaces are nice in the sense that addition and multiplication are imitated by taking direct sums and products. Here, we construct the so-called $sl(2)$-Khovanov homology. Starting the turn-the-crank recipe, we need a few ingredients.

- Let $W = \bigoplus_m W_m$ be a graded vector space with homogeneous components $\{W_m\}$, then the u-graded dimension of W is $\dim_u W = \sum_m u^m \dim W_m$.
- We denote by $\cdot\{l\}$ the 'degree shift' operator on graded vector spaces. This means that $W\{l\}_m = W_{m-l}$ so $\dim_u W\{l\} = u^l \dim_u W$.
- We denote by $\cdot[s]$ the 'height shift' operator on chain complexes. So if $\mathcal{C} = \ldots \longrightarrow \mathcal{C}^r \xrightarrow{d^r} \mathcal{C}^{r+1} \longrightarrow \ldots$, then $\mathcal{C}[s]^r = \mathcal{C}^{r-s}$.

We now define V to be a vector space that is spanned by two elements $v_\pm$ of opposite degree, for which we have $\dim_u V = u + u^{-1}$. Then given any smoothing S_α as before, we assign to it the graded vector space $V_\alpha(L) = V^{\otimes k}\{r\}$. With this, we define the vector space $[L]^r$ to be the direct sum of all the $V_\alpha(L)$ at height r, or $[L]^r = \bigoplus_{\alpha,|\alpha|=r} V_\alpha(L)$. Note that we have $\dim_u V^{\otimes k}\{r\} = u^r(u+u^{-1})^k$. We now have a long sequence

$$[L] = [L]^0 \longrightarrow [L]^1 \longrightarrow \ldots \longrightarrow [L]^n, \tag{9.1.3}$$

from which we may define

$$\mathcal{C}(L) = [L][-n_-]\{n_+ - 2n_-\}. \tag{9.1.4}$$

This complex can be endowed with a degree 0 differential d (it does not change the graded dimensions of the spaces $[L]^r$), which is further explained in [22]. Taken this as a given, we conclude that $[L]$ really is a chain complex, so we can speak of its homology. Since the differential has degree 0, the Euler characteristic of the chain complex and the homology are the same, and we find by construction that it equals the unnormalized Jones polynomial

$$\begin{aligned} \hat{J}_L(u) &= \chi_u(\mathcal{C}(L)) = (-1)^{n_-}u^{n_+-2n_-}\chi_u([L]) \\ &= (-1)^{n_-}u^{n_+-2n_-}\sum_\alpha (-1)^r \dim_u [L]^r \\ &= (-1)^{n_-}u^{n_+-2n_-}\sum_\alpha (-1)^r u^r (u+u^{-1})^k. \end{aligned} \tag{9.1.5}$$

Denoting by $\mathcal{H}^r(L)$ the r^{th} homology group of $\mathcal{C}(L)$, the last claim translates into

$$\hat{J}_L(u) = \sum_{r=0}^{n} (-1)^r \dim_u \mathcal{H}^r(L). \tag{9.1.6}$$

The nice thing about Khovanov homology is that the vector spaces $\mathcal{H}$ are bigraded by the height r and the power of u. Hence, we can define the Poincare polynomial of $\mathcal{C}(L)$ by

$$\mathrm{Kh}_L(u,t) = \sum_{r=0}^{n} t^r \dim_u \mathcal{H}^r(L) = \sum_{r,s\in\mathbb{Z}} t^r u^s \dim \mathcal{H}^{r,s}(L) \tag{9.1.7}$$

which by factoring out the u-dependence we explicitly indicated the bigrading of the homology groups $\mathcal{H}^{r,s}(L)$. In the last equality, it is understood that only finitely many terms in the sum are nonzero. It is through this bigrading that Khovanov homology provides a stronger knot invariant than the Jones polynomial.

Khavanov's theorem [23] now states that:

- $\dim_u \mathcal{H}^r(L)$ is a link invariant
- and hence $\mathrm{Kh}_L(u,t)$ is a link invariant which specializes to $\hat{J}_L(u)$ at $t=-1$.

Precisely, to make the connection to the normalized Jones polynomial $J_L(q)$ from table 4, we have

$$\mathrm{Kh}_L(u,-1) = \hat{J}_L(u) = (u+u^{-1})J_L(u^2). \tag{9.1.8}$$

To prove this, it is enough to check the invariance of $[L]$ under the Reidemeister moves, which are the basic 'building blocks' from which every topological change in a link diagram can be built up from.

Going through the recipe outlined above, it is straightforward to find $\mathrm{Kh}_L(u,t)$ for the trefoil knot, namely for $N=2$, the $SL(2)$ Poincaré polynomial is

$$\mathrm{Kh}_{3_1}(u,t) = u^{-3}(1+u^2+(1+tu^4)t^{-3}u^{-6}) = \frac{u^8+u^6+u^4-1}{u^9}. \tag{9.1.9}$$

Normalizing by $u+\frac{1}{u}$, setting $t=-1$ and recalling that $u=q^{1/2}$, this expression specializes to the Jones polynomial for the trefoil in table 4. Some more examples are listed below.

K	$\mathrm{Kh}(u,t)$
Figure-eight	$t^2u^5+tu+u+u^{-1}+u^{-1}+t^{-1}u^{-1}+t^{-2}u^{-5}$
Solomon's seal	$u^{-3}+u^{-5}+t^{-2}u^{-7}+t^{-3}u^{-11}+t^{-4}u^{-11}+t^{-5}u^{-15}$
Stevedore	$t^2u^5+tu+2u+u^{-1}+t^{-1}u^{-1}+t^{-1}u^{-3}+t^{-2}u^{-5}+t^{-3}u^{-5}+t^{-4}u^{-9}$

Table 5: The Poincaré polynomial for Khovanov homology for some simple knots.

The 2 in $sl(2)$ came in through the vector space we assigned to a given smoothing S_α. In general, for each value of N of $sl(N)$, there is an combinatorial algorithm to construct a $\mathbb{Z}\oplus\mathbb{Z}$-graded complex by using *matrix factorizations* [24]. The $sl(N)$ Khovanov homology is again a bigraded homological theory, for which the above constructions follow analogously.

9.2 A gauge theory description of Khovanov homology

The duality between Chern-Simons theory and super Yang-Mills theory can now be used to give a gauge theory description of Khovanov homology. Recall that the mathematical construction is entirely in terms of algebraic relations that do not make it manifestly clear that Khovanov homology is a topological invariant of a given link. Another observation was that Chern-Simons theory computes knot invariants, such as the Jones polynomial. This raises the question: what field theory then computes Khovanov homology, which is the categorification of the Jones polynomial? In this section we shall describe the conjecture: through the bulk-boundary duality, one can use twisted $\mathcal{N} = 4$ super Yang-Mills to compute Khovanov homology.

The basic objective is to figure out how the field theory computes a (co)homological complex that corresponds to the homological complex used in Khovanov homology. By the duality, $G = SU(N)$ SYM on N D3 branes wrapped on Z gives $SU(N)$ Chern-Simons on ∂Z. Since showed that we can embed this system in type IIB superstring theory, we can apply string dualities to the brane setup: first S-duality, then T-duality.

S-duality is a duality of SYM theory that interchanges electric and magnetic charges and can be proved for abelian gauge groups at the level of path integrals, as discussed in [53]. For non-abelian $U(N)$, S-duality is usually argued to hold by using its familiar embedding in superstring theory, as the world-volume theory of a stack of D-branes. For other ADE groups, orientifold constructions are possible. Another way is to use the AdS/CFT conjecture, which also relates $\mathcal{N} = 4$ SYM to the type IIB theory. This is discussed, for instance, in [2].

Performing these dualities will lift SYM on Z to SYM on $Z \times S^1$, on which we can interpret the partition function as a trace in a cohomological complex of physical states. This complex is conjectured to be equivalent to Khovanov homology.

So why do we first apply S-duality? In general, if we apply T-duality on $\mathbb{R}^9 \times S^1$ in the presence of an NS5-brane, the dual geometry is not $\mathbb{R}^9 \times \tilde{S}^1$ again, but rather $\mathbb{R}^6 \times \mathcal{T}$, where $\mathcal{T}$ is the *Taub-NUT-space.** This space is topologically $\mathbb{R}^4$, but has a radially warped metric given by

$$ds^2 = \frac{1}{4}\left(H(r)d\mathbf{r}^2 + H^{-1}(r)(d\psi + \omega \cdot d\mathbf{r})^2\right), \qquad l \in \mathbb{R}_+, \qquad \nabla \times \omega = \text{grad}\frac{1}{r}. \tag{9.2.2}$$

Here $r \in \mathbb{R}_+$ is a radial coordinate, $\mathbf{r} \in \mathcal{T}$, $H(r) = \left(\frac{1}{r} + l\right)$, ω is a 3-vector, $\psi \in S^1$ parametrizes the circle fibers and the curl is taken with respect to the flat metric on $\mathbb{R}^4$.

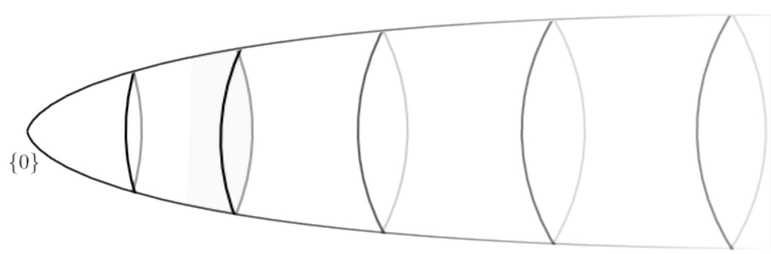


Figure 12: The Taub-NUT space $\mathcal{T}$.

This 4-dimensional space looks like a higher-dimensional cigar and has an S^1-fiber shrinking to zero size at the origin: using quaternions, it is relatively easy to see that if r is small, the metric looks like that of standard spherical coordinates, hence flat. If r is large, the metric looks like that of $\mathbb{R}^3 \times S^1$. Hence,

*From a reverse point of view, T-duality on the Taub-NUT circle fiber (exchanging momentum and winding modes) parametrized by ψ gives a dual circle parametrized by θ, it can be shown [33] that the metric then becomes

$$ds^2_{\text{NS5}} = H(r)\left(d\mathbf{r} \cdot d\mathbf{r} + d\theta^2\right) + \epsilon_{\mu\nu}\omega \cdot \partial^\mu \mathbf{r}\partial^\nu \theta, \tag{9.2.1}$$

which is the metric in the presence of an NS5-brane: the torsion terms gives a non-zero B-field. This is implied by the Busscher rules, which tell us that the Taub-NUT metric is converted into a metric $\tilde{G}$ and non-zero B-field $\tilde{B}$, the latter giving a non-trivial NS5-brane charge.

such a space does not accommodate an interpretation of the path integral as a trace. First applying S-duality takes us to a D3-D5 brane system, upon which T-duality correctly maps us to a geometry with everywhere non-vanishing S^1-fibers (viz. $\mathbb{R}^9 \times \tilde{S}^1$). Details on this can be found in [33, 29].

Applying S-duality: a new look at the Jones polynomial

At this point we want to apply S-duality to topological $\mathcal{N} = 4$ SYM on $Z = M \times \mathbb{R}_-$, the geometric setup discussed above. S-duality maps G to its Langlands dual ${}^L G$, by interchanging the root and coroot lattices. The important example for us is: ${}^L U(N) = U(N)$, since $U(N)$ is the gauge group for the stack of D3-branes. $U(N)$ is not semi-simple, but reductive: it splits as a semi-direct product of the semisimple $SU(N)$ and the abelian $U(1)$, so one can still talk about its root system. Note that the gauge fields on the D3-branes sit in the adjoint representation of $U(N)$, for which the $U(1)$ in $U(N) = SU(N) \times U(1)$ decouples.

Under S-duality, the modular transformation changes the Yang-Mills coupling constant

$$\tau \longmapsto \tau^\vee = -\frac{1}{\mathfrak{n}_\mathfrak{g} \tau}. \tag{9.2.3}$$

Here $\mathfrak{n}_\mathfrak{g}$ is the ratio of long to short roots of $\mathfrak{g}$, so $\mathfrak{n}_\mathfrak{g} = 1$ for simply-laced $\mathfrak{g}$.* Under S-duality, the twisting parameter t is mapped to

$$t^\vee = \pm \frac{\tau}{|\tau|} t = \pm \sqrt{\frac{\tau}{\bar{\tau}}} t \tag{9.2.5}$$

which in combination with (8.3.13) implies $t^\vee = \pm 1$. To preserve chirality, from now on, we choose $t^\vee = +1$.† Since $t^\vee = 1$, the canonical parameter (8.2.7) transforms as

$$\mathbf{k}^\vee = \frac{\theta^\vee}{2\pi} = -\frac{1}{\mathfrak{n}_\mathfrak{g} \mathbf{k}}, \tag{9.2.6}$$

which means that $\mathbf{k}^\vee$ is independent of the coupling parameter $\lambda^\vee$! The instanton winding number term of the dual theory can be defined as

$$W = \frac{1}{2h^\vee} \frac{1}{32\pi^2} \int_Z \mathrm{tr}_{adj}(F \wedge F), \tag{9.2.7}$$

which will weight an instanton by a factor $\exp(-i\theta^\vee W) \equiv q^W$ in the partition function. Here we defined the familiar variable q, for which we have the identification:

$$q = \exp\left(-i\theta^\vee\right) = \exp\left(\frac{2\pi i}{\mathfrak{n}_\mathfrak{g} \mathbf{k}}\right) = \exp\left(\frac{2\pi i}{\mathfrak{n}_\mathfrak{g}(k + h \text{ sign } k)}\right). \tag{9.2.8}$$

Here we made the identification $\mathbf{k} = k + h \text{ sign } k$ analogous from the non-perturbative renormalization effects in Chern-Simons theory. Since $\mathbf{k}^\vee$ is independent of $\lambda^\vee$, we can choose it to be arbitrarily small, so that the partition function Z of the theory localizes on a sum of solutions of the localization equations (8.1.8) at $t^\vee = 1$, which combine to‡

$$\mathcal{U}^+ + \mathcal{U}^- = F - \phi \wedge \phi + * d_A \phi = 0, \qquad * \mathcal{U}^0 = d_A * \phi = 0. \tag{9.2.9}$$

*The S-matrix for this transformation is

$$\pm \begin{pmatrix} 0 & -1/\sqrt{\mathfrak{n}_\mathfrak{g}} \\ \sqrt{\mathfrak{n}_\mathfrak{g}} & 0 \end{pmatrix}. \tag{9.2.4}$$

†The plus or minus sign here depends on applying an additional optional chiral symmetry to map $D3 - NS5$ to the $D3 - \overline{D5}$ system. This will not be of relevance to us here, so we will pick the $+$-sign from now on. Thus, as also discussed in the appendix, S-duality will transform the D3-NS5 system into a D3-D5 system.

‡The index of the Dirac operator (the fermion kinetic term) of the theory calculates the expected dimension of the space of these solutions, which corresponds to the number of zero modes of the Dirac operator. A non-zero index would require extra operator insertions, for convenience, here we assume the index vanishes: this is true at least in the situation that $Z = \mathbb{R}_- \times \mathbb{R}^3$.

Note that these equations are elliptic, as $\mathcal{U}^{\pm}, \mathcal{U}^0$ are. Since $\lambda^\vee$ can be arbitrarily small, localization tells us that the semi-classical approximation is exact and the one-loop (quadratic) approximation to the partition function Z is exact [4]. Z reduces to a ratio of fermion and boson determinants, analogous to section (C.2.2). These determinants are equal up to sign, so every classical solution contributes ± 1 to the partition function. Including the weights given by the instanton winding number term, a classical solution contributes $\pm q^W$. The partition function thus reduces to an index

$$Z_M^{SYM}(q, \{K_i\}) = \int \mathcal{D}X \exp\{Q, Z\} \exp\left(-i\theta^\vee W\right) \xrightarrow{\text{loc}} \text{tr}_{\text{ground states}}(-1)^F q^W = \sum_n a_n q^n \quad (9.2.10)$$

where a_n is the sum of all signs of instantons of winding number $W = n$. The number of such solutions can be expected to be finite, since after imposing elliptic boundary conditions compatible with the K_i, this is an elliptic boundary value problem, which generally admit a finite number of solutions.

One would expect that this generalizes to the case where Wilson loops are inserted on ∂Z, after setting the right boundary conditions on the fields at ∂Z. Then (9.2.10) should represent exactly the Jones polynomial: by the duality between Chern-Simons theory and SYM established previously, we expect that

$$Z_Z^{SYM}(q, \{K_i\}) = Z_{\partial Z}^{CS}(q, \{K_i\}, \mathcal{C}) \quad (9.2.11)$$

on $Z = \mathbb{R}^3 \times \mathbb{R}_-$. Here the left side represents the SYM path integrals after S-duality, that is, they are path integrals with $t^\vee = 1$ and 't Hooft loops inserted on ∂Z, whereas the left hand side is the Chern-Simons path integral over an exotic integration cycle $\mathcal{C}$ with Wilson loops inserted.

The dual boundary conditions

Consider first the case without knots. So what is a suitable elliptic boundary condition? At $s = 0$ a suitable elliptic boundary condition follows from considering (8.1.8). We take $u = 1$ and impose $A|_{\partial Z} = 0, \phi_s|_{\partial Z} = 0$, where $\phi_s|_{\partial Z}$ is the normal component to ∂Z. By rotational and translational invariance in $\mathbb{R}^3 \times \{0\}$, we look for a solution for the tangential part of ϕ that is a function of $s \in \mathbb{R}_-$ only. The localization equation (9.2.9) then reduce to *Nahm's equations*

$$\frac{d\phi^a}{ds} + \epsilon^{abc}\phi^b\phi^c = 0, \qquad a, b, c = 1, 2, 3. \quad (9.2.12)$$

Then a singular solution is simply $\phi^a = t^a/s$, where t^a are generators of the $SU(2)$ Lie algebra that satisfy $[t^a, t^b] = \epsilon^{abc}t^c$; the general solution then is $\phi^a = t^a/s + \ldots$, where the ellipses refer to terms less singular than $\frac{1}{s}$. We impose this boundary condition at $\partial Z = \{s = 0\}$. At $s = -\infty$ the natural choice is that gauge fields go to pure gauge.

In the theory with gauge group G, a link $\cup_i K_i$ is represented by supersymmetric electric Wilson loop operators inserted on $\partial Z = \mathbb{R}^3 \times \{0\}$. After S-duality or electric-magnetic duality, the Wilson loop operators become *'t Hooft operators*, which describe the magnetic field generated as a charge travels along a knot K. 't Hooft operators are determined implicitly by the singularities on their support they create in the worldvolume gauge fields. The insertion of such operators should set the boundary conditions on gauge fields at ∂Z, in such a way that Z equals the same knot polynomial that Chern-Simons theory on ∂Z would compute. However, not much is known about explicit expressions for general G and M, some relevant calculations for $G = SU(2)$ can be found in [29].

At $s = -\infty$, the boundary condition was that the bosonic gauge fields approach a semistable critical orbit of the Morse function $h = \text{Re}\, e^{i\alpha}\mathcal{S}_{CS}$, consisting of flat $G_\mathbb{C}$-connections with $\mu = 0$. Such a flat connection is given by a homomorphism $v : \pi_1(M) \to G_\mathbb{C}$. The dual flat connection should be given by $v^\vee : \pi_1(M) \to G_\mathbb{C}^\vee$, but in general not much is known about $v^\vee$. The trivial case is simple: if $\pi_1(M) = 0$, then $v = v^\vee$ is trivial: this happens exactly in the situation of (8.4.2), where $M = \mathbb{R}^3$.

Assuming $v^\vee$ can be found, computing the Jones polynomial by counting instantons through (9.2.9) in the S-dual picture can be interpreted as a new way of verifying electric-magnetic duality. More explicit calculations have been carried out already in [34] who found some agreement with this and the above statements.

Since knot polynomials are Laurent polynomials, only a finite number of a_n should be non-zero, in the case that $\partial Z \neq \emptyset$. This can only be confirmed on heuristic and experimental grounds so far. Furthermore, in general we might run into framing anomalies, which means that in general we need to add to the action a gravitational term consisting of the Chern-Simons action for the spin-connection on Z, entirely analogous to the discussion in chapter 3.

Applying T-duality: a new view on Khovanov homology

After the previous section, we formally gave a new way to calculate knot polynomials in the world-volume gauge theory for a D3-D5 system. We now want to find a natural cohomological complex that we can interpret as Khovanov homology. For this, we need to interpret the partition function $Z_Z^{SYM}(q, \{K_i\})$ as an Euler characteristic of a cohomological complex. It is clear that to do so, we need to add a dimension to the system: to get such an interpretation as a quantum mechanical trace, we need a circle S^1 in our geometry, as we discussed in chapter 3. $\mathcal{N} = 4$ SYM can conveniently be thought of, in field theoretic terms, as the dimensionally reduced version of 5d $\mathcal{N} = 2$ SYM on $Z \times S^1$. The 4-dimensional topological supercharge Q lifts to a 5-dimensional topological supercharge Q, so that we also have that topological $\mathcal{N} = 4$ SYM lifts to topological $\mathcal{N} = 5$ SYM on $Z \times S^1$. Although this holds for any 4-dimensional manifold Z, we will from now on set $Z = M \times \mathbb{R}_-$.

In a nutshell, the idea behind the main conjecture is the following: we can pick a point $p \in S^1$ and perform quantization on $Z \times p$ from which we get a Hilbert space of physical states $\mathcal{H}(Z)$. The path integral on $Z \times S^1$ then is a trace in $\mathcal{H}(Z)$. Q acts on $\mathcal{H}(Z)$ and by nilpotency of Q we may denote its cohomology as $\mathcal{K}(Z)$, which corresponds to the space of quantum ground states.

Conjecture: the Q-cohomology $\mathcal{K}(Z)$ is equivalent to Khovanov homology.

From the brane perspective, we apply T-duality in the directions transversal to the branes in the D3-D5 system, which lifts us to a D4-D6 system, where the world-volume theory of the D4-branes is twisted 5d SYM. Geometrically, we compactify one macroscopic direction orthogonal to Z: $T^*Z \times \mathbb{R}^2 \longrightarrow T^*Z \times \mathbb{R} \times S^1$, so we can apply T-duality on the S^1.

So what are arguments to support this conjecture? The main argument is that one can show that the Euler characteristic of $\mathcal{K}(Z)$ computes the Chern-Simons partition function and correlation functions, moreover, the geometry naturally supports such a computation. We expound on this below. Secondly, the improved richness of Khovanov homology come from the fact that it uses a $\mathbb{Z} \oplus \mathbb{Z}$ bigraded homological complex. Analogously, states in $\mathcal{K}(Z)$ should sit in representations of some $U(1) \times U(1)$ symmetry. These are provided by the instanton number, the operator W from (9.2.7), which on $Z \times S^1$ should be interpreted as an operator on $\mathcal{K}(Z)$. The other $U(1)$ is furnished by the residual R-symmetry (8.2.2) of the twisted $\mathcal{N} = 4$ theory* : we call the generator of this $U(1)$ symmetry F and so at least $\mathcal{K}(Z)$ is properly bigraded by W and F.

We already explained formally how to calculate $\mathcal{K}(Z)$ in section C.2.3. Consider first the 5-dimensional equations for unbroken supersymmetry in twisted 5d $\mathcal{N} = 2$ SYM, which read

$$F^+_{\mu\nu} - \frac{1}{4} B \times B - \frac{1}{2} D_y B = 0, \qquad F_{y\mu} + D^\nu B_{\nu\mu} = 0. \tag{9.2.13}$$

In the time-independent case, these equations are equivalent to the localization equations (9.2.9) for the 4-dimensional twisted SYM theory, for a proof, see [29]. Hence, solutions to (9.2.9) correspond exactly to

*In the case of general Z, there is no residual $SO(2)$ after a lift to 5-dimensional SYM, whose R-symmetry group is reduced to $SO(5)$. However, upon choosing $Z = M \times \mathbb{R}_-$, we introduce an extra flat direction, which gives us again a residual $SO(2)$ R-symmetry. For general Z however, there always is a $\mathbb{Z}_2$-grading given by the fermion number, of which our F can be seen as a generalization.

classical ground states of the 5d theory.*

Under the assumption of non-degeneracy, every solution to (9.2.9) contributes one ground state to the space of classical ground states $\mathcal{K}_{cl}(Z)$ of the 5-dimensional theory. Recall that the Morse function Re $\mathcal{S}_{CS}$ for Chern-Simons theory is perfect and is equivalent to (9.2.9). Therefore, there are no instanton corrections, and all approximate ground states are exact ground states, that is: $\mathcal{K}_{cl}(Z) = \mathcal{K}(Z)$.

Knowing $\mathcal{K}(Z)$, we can compute its Euler characteristic

$$\chi(q,t) = \mathrm{tr}_{\mathcal{K}(Z)}\left(q^W t^F\right), \tag{9.2.14}$$

in terms of two formal variables q, t. This trace should be expressed as a 5d SYM path integral on $Z \times S^1$, as we recall that we need an S^1 in the geometry to compute traces. There, we may compute the partition function of the space of all physical states $\mathcal{H}$

$$\mathrm{tr}_{\mathcal{H}}\left(q^W t^F \exp\left(-\beta H\right)\right) \tag{9.2.15}$$

where β is the circumference of the S^1, which we should view as the compact imaginary time direction after Wick rotation. A supersymmetric pair with non-zero energy contributes a term

$$q^n \exp\left(-\beta E\right)\left(t^f + t^{f+1}\right)$$

to the partition function. So in order to let only the ground states contribute to this expression, we need $t^f + t^{f+1} = 0$, which implies we should set $t = -1$. Only the expression $E(q, -1)$ can be a topological invariant. By the duality between Chern-Simons theory and topological $\mathcal{N} = 4$ SYM on Z, we must have

$$Z_M^{CS}(q) = \chi(q, -1) = \mathrm{tr}_{\mathcal{K}(M \times \mathbb{R}_-)}\, q^W (-1)^F, \tag{9.2.16}$$

where the expression on the left-hand side is the Chern-Simons path integral with an exotic integration cycle. On $Z = \mathbb{R}^3 \times \mathbb{R}_-$ with knots on $\mathbb{R}^3$, $Z_{\mathbb{R}^3}^{CS}(q)$ computes knot polynomials, as we explained in chapter 7. We see that it would a natural interpretation that the right-hand side is the Euler characteristic at $t = -1$ of Khovanov homology.

To actually prove this, one would have to show that this prescription gives the same calculational rules as the algebraic description of Khovanov homology. The gauge theory proposal has the virtue of making topological invariance manifest, but calculational principles non-trivial. This is the opposite of the situation in the algebraic picture. One of the open issues are the dual boundary conditions: as we remarked earlier, it is not clear in general how to calculate S-dual flat $G_{\mathbb{C}}^{\vee}$-connections $v^{\vee}$. This is not an issue on $Z = \mathbb{R}^3 \times \mathbb{R}_-$ where there are no non-contractible loops. This choice is actually convenient, since Khovanov homology has only been defined for $M = \mathbb{R}^3, S^3$, however, a more general picture is lacking. Moreover, the boundary conditions at $s = 0$ are not known exactly for general gauge groups: the singular behavior of gauge fields at ∂Z is not yet complete understood.

What is tempting though, is that one could have chosen M to be any non-compact simply-connected 3-manifold. In that case, one would only have to contend with determining the right boundary conditions on ∂Z. Assuming this could be done and the conjecture holds, this would significantly generalize the definition of Khovanov homology to a wide array of manifolds, a good improvement over just $M = \mathbb{R}^3, S^3$.

*By analogy, recall that in the (gauged) Landau-Ginzburg model classical ground states correspond to critical points of the Morse function h, which is just the superpotential. After localization, supersymmetric theories localize to the fermionic Q-fixed points, which for supersymmetric quantum mechanics can be found through (??). Looking at the time-independent versions of these fermionic Q-fixed points, these equations simply imply $\frac{\partial h}{\partial \phi^i} = 0$, which is exactly the condition for a classical ground state; that is, the flow has to start at a critical point of h.

10

The road ahead

In this chapter we will discuss consequences and open issues related to the field-theoretic duality we discussed in the previous chapters. We start by a discussion of what the new duality tells us about the consequences of S-duality for Chern-Simons theory. As we will see, this is linked to the meager understanding of Chern-Simons with non-compact gauge group. Secondly, we will discuss how to lift the construction of chapter 8 to M-theory and a recent constructive proposal for a gauge theory description of the Poincare polynomial $\mathrm{Kh}(q,t)$ of (9.1.7). A basic reference for M-theory is [2].

10.1 Modularity and S-duality in Chern-Simons theory

Knot invariants as modular forms

In chapter 8 we applied S-duality to twisted $\mathcal{N}=4$ SYM: the modular S-transformation as usual inverted the SYM coupling constant for simply-laced G as

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \to \tau^\vee = -\frac{1}{\tau}. \tag{10.1.1}$$

and mapped G to its Langlands dual $G^\vee$. As we saw in equation (9.2.10), upon localization this theory is supposed to calculate knot polynomials $Z_M^{CS}(q) = Z_M^{SYM}(q) = \sum_n a_n q^n$ where we define $q = \exp \hbar$. For the moment, we will assume that we normalized the unknot to have knot invariant 1. As we discussed, for compact G, these knot polynomials are Laurent polynomials: they contain a finite number of nonzero a_n. By writing $q^\hbar$ we therefore identified that $\hbar = \frac{2\pi i}{k+h}$, where k is the usual Chern-Simons level and h is the dual Coxeter number of G. Under the modular transformation, one would expect that the S-dual version of SYM with coupling constant $\hbar^\vee$ should compute exactly the same knot polynomials, that is

$$Z_M^{SYM}(K,q) = Z_M^{SYM}(\tilde{K},q^\vee), \tag{10.1.2}$$

since SYM is supposed to be self-dual under S-duality. Here we made explicit that in the S-dual theory, the knot is represented by a 't Hooft loop, the S-dual of the Wilson loop. Now dually, we expect that Chern-Simons theory has a symmetry that maps $\hbar \to -\frac{4\pi^2}{\hbar}$ and G to $G^\vee$.

In more mathematical terms, one therefore expects that $Z_M(K,q)$ should be a *modular form*, that is, an (infinite) q-series that is invariant under the group of modular transformations $SL(2,\mathbb{Z})$. Explicitly, a classic modular form is a holomorphic function f on the complex upper half plane satisfying

$$f_k|_\gamma(z) = (cz+d)^k f\left(\frac{az+b}{cz+d}\right) = f_k(z), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}). \tag{10.1.3}$$

The study of modular forms provides an array of more weaker types of modular forms, such as *mock modular forms, quantum modular forms*, which have variously nice behavior under modular transformations. However, it is immediately clear that for compact G, such nice modular behavior of $Z_M(K,q)$ is not present: since $k+h \in \mathbb{Z}$, under a modular S-transformation,

$$q = \exp \frac{2\pi i}{k+h} \to \exp -2\pi i(k+h) = 1. \tag{10.1.4}$$

Hence, it is impossible to study modular behavior for compact G.

Therefore, one should upgrade to non-compact G, for instance by complexifying G to $G_{\mathbb{C}}$, such that $k+h$ is not necessarily an integer anymore, so q is not an integer root of unity anymore. One would then expect that generically $G_{\mathbb{C}}$ knot polynomials are infinite q-series, which might have nice modular properties. One should interpret such q-series as an extension of the finite knot invariants for compact G, they truncate to the knot invariants for compact G when q is an integer root of unity. However, few systematic calculations are known in Chern-Simons theory for non-compact G.

Experimental examples

In the mathematical literature some 'experimental' work has lead to some specific results in this area. For instance, in [35] the normalized $SU(2)$ Witten-Reshetikhin-Turaev invariant $W(x)$ of the Poincaré homology sphere $\Sigma(2,3,5)$ was studied, defined by

$$W(e^{2\pi i/k}) = e^{2\pi i/k}(e^{2\pi i/k}-1)Z_k(\Sigma(2,3,5)) = e^{2\pi i/k}(e^{2\pi i/k}-1)\int \mathcal{D}A \exp\frac{\tilde{k}}{4\pi}\mathrm{CS}(A), \;\; k=\tilde{k}+2. \tag{10.1.5}$$

The Poincaré homology sphere can be obtained from S^3 by surgery around a $(2,-3)$ torus knot with 2 Dehn twists. Defining

$$A(q) = \sum_{n=1}^{\infty} a_n q^{(n^2-1)/120} = 1+q+q^3+q^7-q^8-q^{14}-q^{20}-\dots, \qquad |q|<1, \tag{10.1.6}$$

where a_n is defined as $a_n = (-1)^{[n/30]}, n^2 = 1 \bmod 120, 0$ otherwise, it was proven that

$$1-\frac{1}{2}A(q) = W(e^{2\pi i/k}), \qquad \text{as } q \to e^{2\pi i/k}. \tag{10.1.7}$$

The proof is quite technical and relies on sophisticated manipulation of algebraic identities. Identity (10.1.7) therefore gives an explicit example of a q-series that truncates at a root of unity to the $SU(2)$ WRT-invariant, but is well-defined when q is not a root of unity. Moreover, it was found that $A(q)$ is not precisely a modular form, but $\tilde{\Theta}_+(q) = q^{1/120}A(q)$ can be massaged into an 'almost' modular form at rational points in the complex lower half plane $\mathbb{H}_-$. We also define $\tilde{\Theta}_-(q) = q^{1/120}\sum_{n=1}^{\infty} b_n q^{n^2/120}$, where $b_n = (-1)^{[n/30]}, n^2 = 49 \bmod 120, 0$ otherwise. To sketch the idea: one can define an auxiliary function $\Theta^*(q)$ that agrees with $\tilde{\Theta}(q)$ at rational points q in $\mathbb{H}_-$. Under a modular transformation $\gamma \in SL(2,\mathbb{Z})$, one has then the equation

$$\left(\frac{cz+d}{-i}\right)^{-1/2}\begin{pmatrix}\Theta_+^*(\gamma(z))\\ \Theta_-^*(\gamma(z))\end{pmatrix} + M_\gamma\begin{pmatrix}\Theta_+^*(z)\\ \Theta_-^*(z)\end{pmatrix} = M_\gamma\begin{pmatrix}R_+\\ R_-\end{pmatrix}, \qquad \gamma = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in SL(2,\mathbb{Z}). \tag{10.1.8}$$

Here M_γ is an arbitrary matrix in $GL(2,\mathbb{C})$ depending on γ and $R_\pm$ are further auxiliary analytical functions. The function $\Theta_\pm^*(z)$ are analogues of the so-called classical Eichler integrals* and are nearly modular with weight $1/2$: the discrepancy is given by the $R_\pm$. The highly non-trivial behavior in (10.1.8) has lead to calling $A(q)$ a *quantum modular form* in [36]: they are functions that are almost modular, up to some 'nice' auxiliary terms.

As another non-trivial illustration, the N-colored Jones polynomial $\mathcal{J}_N(K)$ for the torus knot $T(p,q)$, setting $q=e^\hbar$, is given by

$$2\sinh(N\hbar/2)\frac{\mathcal{J}_N(K)}{\mathcal{J}_N(\bigcirc)} = \exp\left(-\frac{\hbar}{4}\left(\frac{p}{q}+\frac{q}{p}\right)\right)\sum_{\epsilon=\pm1}\sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} \epsilon\exp\left(\hbar pq\left(k+\frac{p+\epsilon q}{2pq}\right)^2\right). \tag{10.1.9}$$

* Given a modular form $f(z) = \sum_{n=1}^{\infty} a_n q^n$ of modular weight ≥ 2, its Eichler integral $\tilde{f}$ is the $k-1$ primitive of f: $\tilde{f}(z) = \sum_{n=1}^{\infty} n^{-k+1}a_n q^n$. The $\tilde{\Theta}_\pm^*(z)$ are then given by $\Theta_\pm^*(z) = \sqrt{\frac{2i}{15}}\int_{\bar{z}}^{\infty}\frac{\Theta_\pm(\tau)d\tau}{\sqrt{\tau-z}}, z\in\mathbb{H}_-$, where $\Theta_+(z) = \frac{1}{2}\sum_{n=1}^{\infty} na_nq^n, \Theta_-(z) = \frac{1}{2}\sum_{n=1}^{\infty} nb_nq^n$.

This gives a definition of the N-colored Jones polynomial at arbitrary q. One can show that at $q \to e^{2\pi i/N}$, this formula reduces to the standard N-colored Jones polynomial. A special feature of this formula is that it can be exactly related to the Eichler integral of the character of the minimal model $M(p,q)$ at $q \to e^{2\pi i/N}$.* , the explicit calculation can be found in [37]. The starting point in the calculation is the partition function or character of $M(p,q)$ given by

$$\mathrm{ch}(p,q,n,m;\tau) = \mathrm{tr}\, q^{L_0 - \frac{1}{24}c(p,q)} = \frac{q^{\Delta(p,q,n,m) - \frac{1}{24}c(p,q)}}{(q)_\infty} \sum_{k\in\mathbb{Z}} q^{pqk^2} \left(q^{k(pn-qm)} - q^{k(pn+qm)+mn} \right), \tag{10.1.11}$$

where $1 \leq n \leq p-1, 1 \leq m \leq q-1$ label the irreducible highest weight representations with conformal weight $\Delta(p,q,n,m)$ of the Virasoro algebra and $q = \exp 2\pi i\tau$. Its Eichler integral at the special values $(n,m) = (s-1,1)$ is

$$\Phi^{(p-1,1)}(1/N) = \frac{pq}{N} \exp\left(\frac{pq}{2} N\pi i + (p+q)\pi i \right) \sum_{\epsilon=\pm 1} \sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} \epsilon \left(k + \frac{p+\epsilon q}{2pq} \right)^2 \exp\left(\frac{2\pi i}{N} pq \left(k + \frac{p+\epsilon q}{2pq} \right)^2 \right), \tag{10.1.12}$$

which equals (10.1.9) in the limit that $\hbar \to \frac{2\pi i}{N}$ becomes an integer root of unity. The details in between again rely on non-trivial algebraic manipulations.

For a slightly more physical picture, modular behavior of the $SL(2,\mathbb{C})$ Chern-Simons partition function was further analyzed on hyperbolic 3-manifolds in [38], where the modular behavior eventually was traced back to the modular transformation properties of the quantum dilogarithm function. However, what remains highly unclear from these mathematical results is how this modular behavior is reflected in physical Chern-Simons theory in general.

Modularity in $SL(2)$ Chern-Simons

A more complete explanation and physical interpretation of modularity in Chern-Simons theory has been provided in [39], where it was shown that $SL(2)$ Chern-Simons theory indeed has a modular symmetry under $\hbar \to -\frac{4\pi^2}{\hbar}$. Explicitly, by realizing Chern-Simons theory through a compactification of a 6-dimensional theory, an explanation for modularity could be given in terms of mirror symmetry on the Hitchin moduli space.

The idea is to consider a system of M5-branes on $M \times S^3$, which can be compactified in two ways: on M or on S^3 plus something extra. Setting $M = \mathbb{R} \times \Sigma$, we can compactify on Σ. This choice gives 4-dimensional $\mathcal{N} = 2$ SYM on $\mathbb{R} \times S^3$, whose gauge group and field content is determined by the choice of Σ. On S^3 an additional Ω-deformation is made, which amounts deforming the diagonal metric on S^3. This is worked out in chapter 4 of [40]. The idea is that S^3 has a $U(1) \times U(1)$-action, which can be used to introduce a non-trivial monodromy around S^1-fibers in S^3: locally the fiber bundle does not have a direct product structure anymore.* The Ω-deformation introduces two parameters ϵ_1, ϵ_2 and

*The minimal model CFT $M(p,q)$ is characterized by the fact that every family of Virasoro descendants is finite, which as it turns out, can be characterized by two integers p,q. This is covered in detail in [21]. N-colored means that the Wilson loops sit in the N-dimensional irreducible representation of $SU(2)$. This representation can be constructed for instance by taking all homogeneous polynomials of order $N-1$ in two complex coordinates $\mathbf{z} = (z_1, z_2)$, where the group homomorphism

$$\pi : SU(2) \to \{\text{Homogenous polynomials of order } N-1\} \tag{10.1.10}$$

for $U \in SU(2)$ is given by $\pi(U)f(\mathbf{z}) = f(U^{-1}\mathbf{z})$.

* For S^3, one can use coordinates $y^\mu, \mu = 1, \ldots 4$ in which S^3 is given by $||y||^2 = 1$. In polar coordinates, we can write $y_1 + iy_2 = ue^{i\alpha}, y_3 + iy_4 = ve^{i\beta}, (u,v) = (\cos a, \sin a)$. The $U(1) \times U(1)$-action shifts α, β. An $U(1) \times U(1)$-invariant metric on S^3 is $da^2 + f(a)d\alpha^2 + g(a)d\beta^2$. Introducing two auxiliary angular variables θ_1, θ_2, the Ω-deformed metric is then given by

$$ds^2 = da^2 + f(a)(d\alpha - \epsilon_1 d\theta_1)^2 + g(a)(d\beta - \epsilon_2 d\theta_2)^2 + d\theta_1^2 + d\theta_2^2.$$

one gets a Hilbert space $\mathcal{H}^{\mathcal{N}=2}(\epsilon_1,\epsilon_2,S^3)$ of physical states associated to the $\mathcal{N}=2$ gauge theory on $(\mathbb{R}\times S^3)_{\epsilon_1,\epsilon_2}$. On the other hand, compactifying on S^3 with a topological twist on M gives $SL(2)$ Chern-Simons theory on M in the presence of the Ω-deformation, see [41]. So alternatively there is a Hilbert space $\mathcal{H}^{CS}_{\hbar}(\Sigma)$ associated to quantization of the phase space of Chern-Simons theory: the space $\mathcal{M}_{\text{flat}}(G,\Sigma)$ of flat connections on $\mathbb{R}\times\Sigma$. The statement now is that

$$\mathcal{H}^{\mathcal{N}=2}(\epsilon_1,\epsilon_2,S^3)\cong\mathcal{H}^{CS}_{\hbar}(\Sigma),\qquad \hbar=2\pi i\frac{\epsilon_1}{\epsilon_2}. \tag{10.1.13}$$

From this it is clear that the modular transformation $\hbar\to\hbar^\vee$ corresponds to $\epsilon_1\longleftrightarrow\epsilon_2$. The partition function of the theory can also be obtained from this data, by looking at the mapping class group of Σ: as this is not essential here, however, we will leave those details to [39].

The key idea is that one should study the Hitchin moduli space

$$\mathfrak{H}=\mathcal{M}_H(G,\Sigma) \tag{10.1.14}$$

of solutions to the self-duality equations for gauge theory on a Riemann surface Σ, which are the familiar equations $\mathcal{F}=d_A*\phi=0$. A key feature of $\mathcal{M}_H(G,\Sigma)$ is that it is hyperkähler and that in complex structure J, it is isomorphic to the space $\mathcal{M}_{\text{flat}}(G_{\mathbb{C}},\Sigma)$ of flat $G_{\mathbb{C}}$-connections on Σ.* The details can be found in [17]. Recall from chapter 7 that $\mathcal{M}_{\text{flat}}(G_{\mathbb{C}},\Sigma)$ is the phase space of $G_{\mathbb{C}}$ Chern-Simons theory on any 3-manifold whose boundary is Σ.

Associated to J is a symplectic structure Ω_J, so that we can quantize $(\mathfrak{H},\Omega_J)$. With the observation made above, upon quantization one finds that the Hilbert space of physical states is exactly $\mathcal{H}^{CS}_{\hbar}(\Sigma)$: this tells us that any nice behavior of $\mathcal{H}$ and its quantization under S-duality will be reflected in $\mathcal{H}^{CS}_{\hbar}(\Sigma)$!

As the simplest example, one can now take $\Sigma=T^2$ and $G_{\mathbb{C}}=SL(2,\mathbb{C})$, such that

$$\mathcal{M}_{\text{flat}}(SL(2,\mathbb{C}),T^2)=(\mathbb{C}^*\times\mathbb{C}^*)/\mathbb{Z}_2=(S^1\times\mathbb{R}_+)^2/\mathbb{Z}_2, \tag{10.1.15}$$

given by the complexification of the $U(1)\times U(1)$-rotation group that measures the winding numbers around the non-trivial 1-cycles on T^2 (we'll come back to this below). Note that this space is a toric variety, it can be seen as the total space of a torus fibration. The importance of this example is of course that the knot complement in a simply connected 3-manifold is exactly a manifold with a T^2 as boundary: the torus that surrounds the loop of the knot.

In σ-model language, we can think of the σ-model with target space $\mathfrak{H}$. In this setting, it can be shown that S-duality amounts to mirror symmetry plus an extra hyperkähler rotation:

$$\text{S-duality}\longleftrightarrow\text{ mirror symmetry }\circ\begin{pmatrix}J\to K\\ K\to -J\end{pmatrix}. \tag{10.1.16}$$

This was shown in [17]. Now the mirror of $\mathcal{M}_{\text{flat}}(G_{\mathbb{C}},T^2)$ is well-known: from [42] we learn that it is just $\tilde{\mathfrak{H}}=\mathcal{M}_{\text{flat}}(G^\vee_{\mathbb{C}},T^2)$! In our example, one finds by interchanging roots and coroots that the mirror dual to $SL(2,\mathbb{C})$ is $SO(3,\mathbb{C})$. Note that the hyperkähler rotation does not affect this statement: it only modifies the classification of branes. By our earlier identification, this means that the two Hilbert spaces $\mathcal{H}^{CS}_{\hbar}(T^2)$ and $\mathcal{H}^{CS}_{\hbar^\vee}(T^2)$ are isomorphic. The intuitive idea now is clear: knot invariants are computed in Chern-Simons theory by surgery: this defines algorithmic operations in $\mathcal{H}^{CS}_{\bullet}(\bullet,T^2)$ which amount to the skein relations such as (7.3.20). Hence, one would expect that this isomorphism respects the skein relations and implies modular behavior of knots invariants.

This can be checked. When $\Sigma=T^2$, flat connections are conveniently parametrized by their $\mathbb{C}^*$-valued holonomies around the non-trivial 1-cycles of T^2. Specifically, the holonomies are given by two $\mathbb{C}^*$-valued numbers $m=e^u, l=e^l$ modulo the $\mathbb{Z}_2$ Weyl action $m\to m^{-1}, l\to l^{-1}$. The notation l,m

* The complex structures on $\mathcal{M}_H(G,\Sigma)$ correspond to splitting the three real equations $F-\phi\wedge\phi=d_A*\phi=d_A\phi=0$ into one complex complex and one real equation. All the details can be found in chapter 4 of [17].

refers to the longitudinal and meridian cycle on T^2 respectively. Suppose now that we have a knot K in a simply-connected 3-manifold, where T^2 surrounds the knot K. l and m can be taken as coordinates for $\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, T^2)$, since flat complex connections are uniquely characterized by their holonomies around T^2, or equivalently, K. It turns out that flat connections $\mathcal{A}$ are completely captured by the A-polynomial $A(l,m)$: one has that

$$\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, T^2) = \{\mathcal{A} \mid A(l,m) = 0\}. \tag{10.1.17}$$

Heuristically, the A-polynomial classifies those flat connections around $\Sigma = T^2$ that extend to flat connections in the bulk M.

One can now explicitly construct the action of S-duality on flat complex connections in $M = \mathbb{R} \times T^2$ by studying the A-polynomial. For example, for a (p,q) torus knot in $\mathbb{R} \times T^2$ one finds the A-polynomial $lm^{pq} + 1$. By applying the operations in (10.1.16) one finds that the dual A-polynomial is $A^\vee(l^2, m^2) = A(-l,m)A(l,m)$. This shows explicitly what the dual flat $G^\vee$-connections are.

This story can be generalized to any manifold whose boundary is T^2 and to an arbitrary gauge group, as long as the mirror map holds between $\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, T^2)$ and $\mathcal{M}_{\text{flat}}(G^\vee_{\mathbb{C}}, T^2)$. One can show explicitly that $\mathfrak{H}$ and $\tilde{\mathfrak{H}}$ fiber over a certain base space B and that in general, $\mathfrak{H} = (\mathbb{C}^*)^n \times (\mathbb{C}^*)^n / \mathcal{W}$, where $\mathcal{W}$ is the Weyl group of reflections. Hence mirror symmetry is guaranteed by the SYZ picture of mirror symmetry (see [43]): mirror symmetry amounts to T-duality on the torus fibers over B.

10.2 M-theory and gauge theory dualities

In chapter 8, 5-dimensional $\mathcal{N} = 2$ SYM was central in the gauge theory proposal for Khovanov homology. Recall from (2.1.2) that the field strength is the curvature $F_{\mu\nu} = [D_\mu, D_\nu]$. Since D_μ has dimension 1, A_μ is of dimension 1 too, so by power counting, the Yang-Mills coupling constant g has negative dimension in dimension 5, as it appears inversely in the Lagrangian as $L \sim \frac{1}{g^2} \operatorname{tr} F^2$. Hence the coupling grows under the renormalization group flow: this implies that the theory becomes strongly interacting at high energies and therefore naively is not UV-complete. From a field theory point of view, using this description is therefore slightly dissatisfying.

However there is a more complete M-theory picture, since 5-dimensional $\mathcal{N} = 2$ SYM can be seen as the dimensional reduction of a 6-dimensional $(0,2)$ superconformal field theory, which is UV-complete. In this picture, the D4-branes come from a dimensional reduction of a stack of M5-branes, whose worldvolume theory is the $(0,2)$ SCFT. Few concrete details are known about this theory, in part since so far no Lagrangian description is known. It is possible that therefore, one should think of the M5 branes as a purely quantum object, for which no (semi-)classical description is possible. Despite this incompleteness, one can still describe the gauge theory duality of chapter 8 by starting from an M-theory setting.

The idea is to consider M-Theory on $X \times \mathcal{T}$, where X is 7-dimensional and $\mathcal{T}$ is the 4-dimensional Taub-NUT space. A system of N M5-branes on a 6-dimensional manifold $V \times \mathbb{R}^2 \subset X \times \mathcal{T}$. $\mathbb{R}^2$ inherits from $\mathcal{T}$ a cigar metric:

$$ds^2 = dr^2 + f(r)d\varphi^2, \qquad r \in \mathbb{R}_+, \varphi \in [0, 2\pi]. \tag{10.2.1}$$

where we choose $f(r)$ to be suitably decreasing as $r \to \infty$, such that circles at every $r > r_0$ have the same radius, for some $r_0 > 0$. In this description, there always will be a singular circle fiber at the origin. We can now dimensionally reduce on the circle fibers, which gives a space $\mathbb{R}^2/U(1) \cong \mathbb{R}_+$, where the $U(1)$ represents rotation on the circle fibers. From the 11-dimensional perspective, this dimensional reduction gives type IIA superstring theory on $X \times \mathcal{T}/U(1) \cong X \times \mathbb{R}^3$. The subtlety now is that there is a D6-brane supported on $X \times \{0\} \subset X \times \mathbb{R}^3$, where $\{0\}$ corresponds to the boundary of $\mathbb{R}_+$, the fixed point of the $U(1)$-action. The M5-branes wrapped on $V \times \mathbb{R}^2$ become D4-branes wrapped on $V \times \mathbb{R}_+$, so we obtain exactly the D4-D6 system we obtained after T-duality in section (9.2.3), with the identification:

$$Z = M_3 \times \mathbb{R}_-, \qquad V = M_3 \times S^1, \qquad \mathbb{R}_- \cong \mathbb{R}_+, \qquad V \times \mathbb{R}_+ \cong Z \times S^1 \cong M_3 \times \mathbb{R}_- \times S^1. \tag{10.2.2}$$

10.3 Categorification from Ω-deformations and refined Chern-Simons theory

In chapter 8 we discussed a new proposal for a gauge theory description of Khovanov homology as the space of quantum ground states in 5-dimensional $\mathcal{N} = 2$ SYM in a D4-D6 brane setup, with the D4-branes living on $\mathbb{R}^3 \times \mathbb{R}_+ \times S^1$. We have seen that this system can be lifted to M-theory.

A constructive, more limited, proposal for a gauge theory definition of the Poincaré polynomial has been made in [44]. Recall that the essential feature of Khovanov homology was that it gave us a bigraded structure on the knot invariants, which made it generically stronger than, for instance, the Jones polynomial. In chapter (8) we described a proposal to find directly the bigraded homology groups that give the Poincaré polynomial $\text{Kh}(q,t)$ defined in (9.1.7). The idea in [44] is to directly find $\text{Kh}(q,t)$ from a so-called refinement of Chern-Simons theory, which furnishes an extra grading in the theory.

To understand this setup, we first need to define refined Chern-Simons theory on a special class of 3-manifold M. The to-be-defined refinement will generate an extra grading in the theory. Having set this up, we then explain the relation to Khovanov homology by using the large N dual.

An interlude on the topological string

Recall from our discussion of zero modes in the topological A-model, we found the expression (3.3.9), which gave a selection rule for A-model correlators. A salient detail of this formula is that it shows that even if the target space is Calabi-Yau, still only worldsheets with $g \leq 1$ can contribute to non-zero A-model correlators. There is a straightforward way to remedy this: coupling of the A-model to topological worldsheet gravity. After this coupling, an extra contribution to the index calculation coming from the metric moduli ensures that on Calabi-Yau target spaces, the topological string actually never has an anomaly.

Coupling to topological gravity here means that we want to add a Einstein-Hilbert term $\# \int_\Sigma R$ for the worldsheet metric h. Normally, one would then deal with gauge equivalence given by worldsheet diffeomorphisms by doing the Fadeev-Popov procedure and add an additional path integral over the space of all worldsheet metrics.

This procedure superficially can have issues concerning anomalies of the gauge symmetry of worldsheet diffeomorphisms at the quantum level and subtleties generated by large diffeomorphisms. Concerning the first issue, one finds that there is actually no conformal anomaly in the A-model, as the central charge c always vanishes for the topological A and B-model. This is a simple consequence that at the level of worldsheet currents, the topological twist corresponds to a shift of the stress-energy tensor $T(z) \to T(z) + \frac{1}{2}\partial J(z)$, implying that after twisting its Laurent modes $T(z) = \sum_m \tilde{L}_m z^{-m-2}$ satisfy $\tilde{L}_m = L_m - \frac{1}{2}(m+1)J_m$. Straightforward algebra then shows that there is no central charge term left in the commutator relation for $\tilde{L_m}$:

$$[\tilde{L}_m, \tilde{L}_n] = (m-n)\tilde{L}_{m+n}. \tag{10.3.1}$$

With respect to the second issue, one can show that a worldsheet of genus g has $3(g-1)$ metric moduli. A simple example follows by considering the torus: it has a residual complex modulus τ parametrizing the skewness of the torus seen as a lattice $\mathbb{C}/(\mathbb{Z}\text{Re }\tau \oplus i\mathbb{Z}\text{Im }\tau)$ where $\tau \in \mathbb{C}$. This modulus cannot be fixed by conformal transformations, which by definition preserve angles (recall that we can only use the conformal invariance of the 2-dimensional σ-model (string theory) to fix the metric).

One can now check the structure of the OPEs of the worldsheet currents in the A-model are analogous to that of the bosonic string. This tells us that it is straightforward to couple the topological A (and B)-model to topological gravity: we can just additionally add an integration over the moduli space of worldsheet metrics! This is the definition of the *topological string* [45, 46]. It is then straightforward to show that the total axial R-charge becomes $6(g-1) - 2d(g-1)$, where d is the complex dimension of the target space M. From this we see that the topological string is richest when the target space is a Calabi-Yau 3-manifold. It is a nice coïncidence that these are exactly the spaces that are suitable for superstring compactifications.

Refined Chern-Simons and M-theory

The refinement works in the following way. For any 3-manifold M, $SU(N)$ Chern-Simons theory on M is equivalent to the open A-string on T^*M, with N Lagrangian A-branes on M. Here, the string coupling g_s is related to the Chern-Simons coupling k and dual Coxeter number $h = N$ as

$$g_s = \frac{2\pi i}{k+N}. \tag{10.3.2}$$

It is a mathematical result that there are no holomorphic embeddings of the worldsheet Σ into T^*M, so that only degenerate maps can contribute upon localization for the A-string. One finds in this special case that the constant, degenerate maps, give exactly all the perturbative diagrams of Chern-Simons theory, so that we have

$$Z_M^{CS}(q = e^{g_s}) = Z_{T^*M}^A(g_s). \tag{10.3.3}$$

In this description, adding a knot $K \subset M$ corresponds to adding a non-compact Lagrangian A-brane $\mathcal{L}_K$, which comes with a flat bundle $E \to \mathcal{L}_K$. L_K is chosen such that $L_K \cap M = K$.

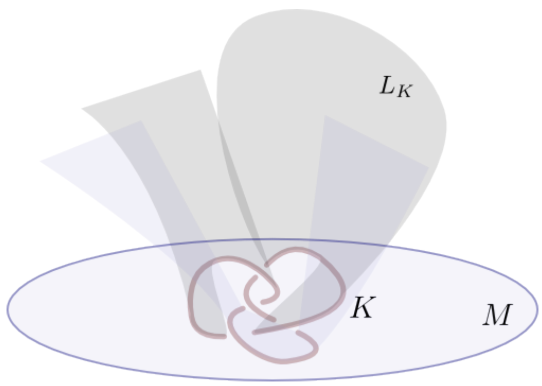


Figure 13: A knot K by intersecting M with a Lagrangian brane L_K.

So how do we get the open topological string from M-theory? Recall that we considered M-theory on $X \times \mathcal{T}$ in section (10.2). We now make a choice for the background as

$$(Y \times \mathcal{T} \times S^1)_q, \tag{10.3.4}$$

where Y is Calabi-Yau. The subscript here indicates that this space is a twisted product: the Taub-NUT space $\mathcal{T}$ is twisted non-trivially around the circle S^1. Here, we can take $\mathcal{T}$ to have the same complex structure as $\mathbb{C}^2$. This twist is specified by defining that by going around the S^1 once, the complex coordinates z_1, z_2 on $\mathcal{T}$ are rotated by $z_1 \to q z_1, z_2 \to q^{-1} z_2$. The partition function of the closed A-string corresponds to the M-theory partition function on this background. Since we're interested in the open A-string, we need to add N M5-branes on $(L \times \mathbb{C} \times S^1)_q$, where L is a Lagrangian submanifold in Y and $\mathbb{C} \subset \mathcal{T}$ (the plane spanned by z_1). The partition function of the M5-branes turns out to be

$$Z_{M5}(Y, L, q) = \text{tr}\left((-1)^F q^{S_1 - S_2}\right) = Z_{open}(Y, L, g_s), \tag{10.3.5}$$

where $S_{1,2}$ are the generators of the $U(1)$ rotations along $z_{1,2}$ and $F = 2S_1$ is the fermion number. The second equality follows by construction.

The twisted space (10.3.4) is a special case of an Ω-background, one says that $(Y \times S^1)_q$ is an Ω-deformed space. The general Ω-background is the case where we twist the circle fiber by saying that going around a circle fiber once, we map $z_1 \to qz_1, z_2 \to t^{-1}z_2$. Obviously, if $t = q$, we specialize to the case above. The resulting space is denoted by

$$(Y \times \mathcal{T} \times S^1)_{q,t}. \tag{10.3.6}$$

The problem is that this choice spoils supersymmetry: recall that any supercharge Q must have half-integer eigenvalues under the $U(1)$-rotations. From (10.3.5), we see that it is possible to have a supercharge Q that has $S_1 - S_2$-eigenvalue 0, such that (10.3.5) defines a good supersymmetric index. However, in the general case, the partition function would be

$$\mathrm{tr}\left((-1)^F q^{S_1} t^{-S_2}\right). \tag{10.3.7}$$

Clearly, no supercharge can have vanishing S_1, S_2-eigenvalues separately: this means that states with non-zero energy will not cancel out in the partition function anymore: supersymmetry is broken. So we need to twist by an extra $U(1)_R$ R-symmetry, to obtain one we need to take the Calabi-Yau Y to be non-compact. On a non-compact Y, the effective 5-dimensional worldvolume theory on Y obtains an extra $U(1)_R \subset SU(2)_R$ symmetry. A similar twist is needed when adding M5-branes on $Y = (M \times \mathbb{C} \times S_1)_{q,t}$. This puts an extra constraint on M, one can show that in the generic case M has to be a Seifert manifold. A Seifert manifold is an S^1 fibration over a genus g Riemann surface, the $U(1)$-action being provided by the rotation of the fiber. Note that at some points, one can have a discrete stabilizer, but this is well-behaved with respect to the twisting, for the same reason in (5.1.2). The most simple example is S^3, which is Seifert by viewing it as the Hopf fibration.*

After twisting, one defines the partition function

$$Z_{open}(T^*M, q, t) = \mathrm{tr}\left((-1)^F q^{S_1 - S_R} t^{-S_2 + S_R}\right) = Z_{CS}(M, q, t), \tag{10.3.8}$$

where the final term is the partition function of the refined Chern-Simons theory. We can now have a supercharge with eigenvalues $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ under (S_1, S_2, S_R), which restores supersymmetry.

If we add a Wilson loop on a knot $K \subset M$, the knot insertion has to respect the extra R-symmetry on the Seifert manifold M, such that the twist remains valid. It follows that this means the Wilson loop has to be inserted on the S^1 fibers on M. An intuitive reason is that in that case, the $U(1)_R$-orbit is the knot K.

The refined theory is not an alternative definition of Chern-Simons theory, as the Lagrangian is unchanged under the refinement. However, the coupling parameters of the theory are redefined, the knot invariants computed by the refined theory now are function of q, t, instead of just q. The computation of knot invariants in the refined theory is completely analogous to the way knot invariants are computed in the unrefined theory, namely by exploiting gluing and pasting operations on the knot K and the background M as in section 7.3.

Large N dual

In the previous section we defined in (10.3.8) the partition function of refined Chern-Simons theory by exploiting the link to M-theory. The question now is why this expression should match the Poincaré polynomial $\mathrm{Kh}(q, t)$ from (9.1.7). It is argued in [] that to make such an identification, one should at least identify a relation between q and t by going to the large N dual of refined Chern-Simons theory.

Let us first discuss this for the unrefined case. In the two papers [47, 25] a physical interpretation of Khovanov homology was given. The first starting point in [47] is the idea that for unrefined Chern-Simons theory, its large N dual is known. Namely, $SU(N)$ Chern-Simons on S^3 has a large N dual given by the

* Identify $\mathbb{R}^4 \cong \mathbb{C}^2$. Then S^3 is the locus of $|z_1|^2 + |z_2|^2 = 1$ and S^2 is the locus $|z_1|^2 + (\mathrm{Re}\, z_2)^2 = 1$. Define $\pi(z_1, z_2) = 2z_1 z_2^*, |z_1|^2 - |z_2|^2)$, then it is easy to check that $\pi(z_1, z_2)$ lies on S^2 in $\mathbb{C} \times \mathbb{R}$ when $(z_1, z_2) \in S^3$. Now note that if $\pi(z_1, z_2) = \pi(w_1, w_2)$ iff $(w_1, w_2) = \lambda(z_1, z_2), |\lambda|^2 = 1$. Hence the inverse image of $\pi^{-1}(x)$ is a circle for all $x \in S^2$ and S^3 is a disjoint union of all these fibers.

closed topological string on the conifold $X = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{CP}^1$.* N is then related to the size of $\mathbb{CP}^1 \subset X$.

In [44], it is argued that a similar large N dual can be found for the refined version: its dual should be the refined A-string. Using the same M-theory duality as explained above, leads to the conjecture that the Poincaré polynomials of $SL(N)$ Khovanov homology are computed by refined Chern-Simons theory. This conjecture has only been been confirmed in a few simple cases, moreover it is limited to the case where M is Seifert and torus knots are inserted.

So how does this work? Suppose that no knots are inserted on S^3. Then using the notation as above, the dual theory at large N should be the refined closed topological string on $X = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{CP}^1$. One can show that the partition function in this background is

$$Z_{closed}(X, \lambda, q, t) = \exp\left(-\sum_{n=0}^{\infty} \frac{\lambda^n}{n\left(q^{n/2} - q^{-n/2}\right)\left(t^{n/2} - t^{-n/2}\right)}\right), \tag{10.3.10}$$

where $\lambda = \exp\left(-\text{Area}\ \mathbb{P}^1\right) = q^N = q^{g_s N}$. From cutting and gluing procedures like those in (7.3), one has that the refined Chern-Simons theory has partition function

$$Z_{CS}(S^3, q, t) = S_{00} = \prod_{m=0}^{\beta-1}\prod_{i=1}^{N-1}(1 - t^{N-i}q^m)^i \to \prod_{m=0}^{\beta-1}\prod_{i=1}^{\infty}(1 - t^{N-i}q^m)^i, \tag{10.3.11}$$

where we identify $t = q^\beta, \beta \in \mathbb{N}$ and we took the $N \to \infty$ limit. Here S_{00} is a matrix element in the group $SL(2, \mathbb{Z})$ of large diffeomorphisms of T^2. The last expression works out to be exactly (10.3.10) by simple manipulations and another identification: $\lambda = t^{N+1/2}q^{-1/2}$.

Note however, that this argument uses the large N duality; all knot polynomials that are computed and have been checked use small N, typically $N = 2, 3, \ldots$. Therefore, the agreement in calculations obtained in [44] is surprising.

A conjecture

The concrete proposal to compute $\text{Kh}(q, t)$ in refined Chern-Simons theory now is the following. By using the identifications between q, t, λ quoted earlier, one can compute by surgery (as in the unrefined case) the normalized knot invariant $Z(S^3, K)/Z(S^3, \bigcirc)$, where $\bigcirc$ is the unknot. Now we needs to set

$$a = \sqrt{t}, \qquad b = -\sqrt{q/t}, \qquad c = \sqrt{\lambda}, \tag{10.3.12}$$

so that

$$Z(S^3, K)/Z(S^3, \bigcirc) = f(a, b, c). \tag{10.3.13}$$

Note that upon setting $t = -1$ and $c = q^N$, one gets the $SL(N)$ HOMFLY polynomial. The conjecture is that

$$f(q, c, t) = \sum_{i,j,k} t^i q^j c^k \dim H^{i,j,k}(K), \tag{10.3.14}$$

which upon setting $c = q^N$ computes the $SL(N)$ Poincaré polynomial $\text{Kh}(q, t)$. Here $H^{i,j,k}(K)$ are the homology groups that categorify the $SU(N)$ knot invariants, the bigraded HOMFLY polynomials. For

* The weighted bundle $\mathcal{O}(n_1) \oplus \ldots \oplus \mathcal{O}(n_p) \to \mathbb{P}^q$ is defined as the space $\mathbb{C}^{p+q+1}/(\mathbb{C}^p \times \{0\}_{q+1})$ with identification of coordinates given by

$$(z_1, \ldots, z_p, w_1, \ldots, w_{q+1}) \sim (\lambda^{n_1} z_1, \ldots, \lambda^{n_p} z_p, \lambda w_1, \ldots, \lambda w_{q+1}) \tag{10.3.9}$$

for $\lambda \neq 0$. The conifold comes from the $S^3 \to S^2$ geometric transition: the conifold singularity $||y||^2 = 0, y \in \mathbb{C}^4$ can be described as an S^2-bundle, since one can rewrite the defining equation as $ab - cd = 0 \implies a = \lambda d, c = -\lambda b; a, b, c, d \in \mathbb{C}$. Taking $\lambda \in S^2 \cong \mathbb{CP}^1$, this equation is the defining equation for the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to S^2$. A resolution of the singularity can be given in two ways: one can deform to the total space of T^*S^3 or to that of the quoted bundle.

$N = 2$, these specialize to the case of the Jones polynomial and its categorification, Khovanov homology.

Going through this procedure, it was checked in [44] that one gets exact agreement for $\mathrm{Kh}(q, t)$ (see also table 5) for the trefoil and torus knots of type $(2, 2m - 1)$. For other knots, it is less straightforward to determine the surgery matrices S, T from section 7.3.2 that facilitate the computations, those cases remain to be checked against this conjecture.

Physical interpretation of Khovanov homology

Conjecture (10.3.14) is based on a conjecture for a physical interpretation of knot categorifications which was posited in [47] and extended in [25]. We cited earlier the large N dual of unrefined Chern-Simons theory. Moreover, we saw the duality of the A-string on X with M-theory on $(X \times \mathcal{T} \times S^1)_q$. Using these two dualities one can interpret the conclusion in [47] by saying that knot invariants on S^3 can be computed by counting BPS states* of the M2-M5 brane system on $(X \times \mathcal{T} \times S^1)_q$, where the M5-branes are wrapping L_K. That is, one has

$$Z^{CS}(K, S^3, V, q) = Z_{M2-M5}(L_K, X, V, q). \tag{10.3.15}$$

In the intermediate step between Chern-Simons and the topological string, the relation between $SL(N)$ Chern-Simons knot invariants and the topological string was conjectured to be

$$J_N(q) = \frac{1}{q - q^{-1}} \sum_{s,Q \in \mathbb{Z}} N_{\square,Q,s} q^{NQ+s}, \tag{10.3.16}$$

where the integers $N_{\square,Q,s}$ count the number of BPS states in the string Hilbert space. The subscript $\square$ indicates that we take the Wilson loops in the fundamental representation $\square$ of $SL(N)$. The main point of the right-hand side of (10.3.15) is that it is computed by a trace in a triply-graded vector space $\mathcal{H}^{S_1,S_2,Q}_{BPS}(L_K)$, the space of BPS states in the M2-M5 brane theory. In the M-theory picture, the grading is provided by the two generators $S_{1,2}$ of $U(1)$-rotations on $\mathcal{T}$ as before and the M2 brane charge $Q \in H_2(X, \mathbb{Z})$. Again from the M-theory perspective, [25] conjectured that the $\mathcal{H}^{S_1,S_2,Q}_{BPS}(L_K)$ are isomorphic to the triply-graded homology groups $H^{i,j,k}(K)$. From the topological string perspective, the correspondence at the level of Poincaré polynomials was conjectured to be

$$(q - q^{-1})\mathrm{Kh}(q, t) = \sum_{Q,s,r \in \mathbb{Z}} D_{Q,s,r} q^{NQ+s} t^r, \tag{10.3.17}$$

where the integers $D_{Q,s,r}$s are defined by

$$N_{\square,Q,s} = \sum_{r \in \mathbb{Z}} (-1)^r D_{Q,s,r}. \tag{10.3.18}$$

Although this conjecture has not been proved rigorously yet, it can be checked in the affirmative for a few simple knots, as has been done in [25].

* BPS states sit in short supermultiplets that arise by considering the case with $Z^{AB} \neq 0$ in (2.2.1). Depending on the values of the eigenvalues of Z^{AB} it is straightforward that less of the supercharges satisfy a fermionic oscillator algebra; recall if they do they can be used to construct raising and lowering operators. Hence, in that case, the admissible supermultiplets become shorter. In general, D-branes break part of the space-time translation invariance, since the open superstring only has $\mathcal{N} = 1$ worldvolume supersymmetry. Hence D-branes break supersymmetry partially, and hence can be viewed as BPS states in the type IIB theory. More details can be found, for instance, in [2].

11

Conclusion and outlook

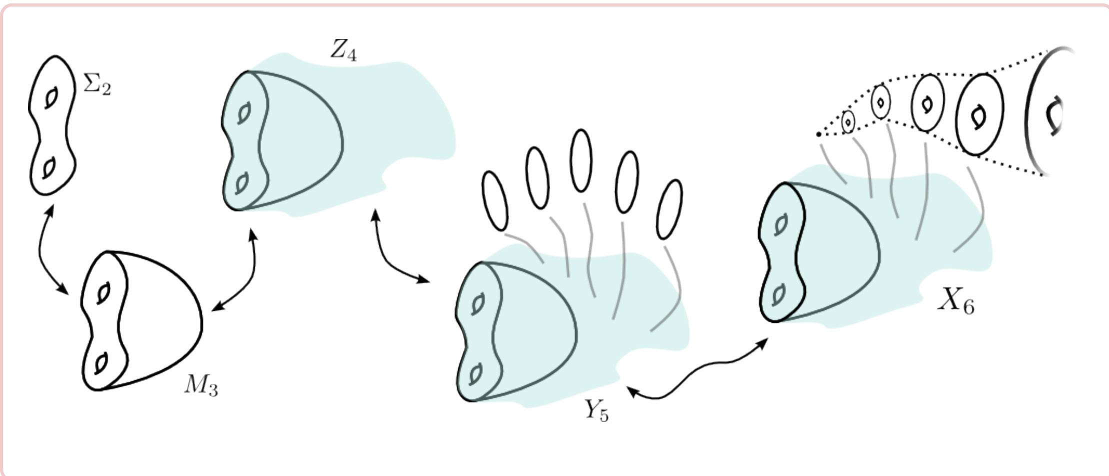
Summary

At this point, we have reviewed several topics in mathematical physics. In the first half of this thesis, we discussed supersymmetric gauge theories and supersymmetric localization. We then discussed topological supersymmetry, in order to define supersymmetry on curved manifolds. The open and closed A-model and topological branes were covered at length, together with their categorical descriptions.

We then continued by discussing an application of Morse theory to field theory: it gave a way to re-express path integrals by using Morse theory after complexifying the source theory. We then applied this technique to quantum mechanics and showed what the technical subtleties were in applying this to the simple harmonic oscillator. Details on a new view on quantization by using the A-model were also given and we showed how this applies to the SHO.

After explaining how Chern-Simons theory computes knot invariants, we applied the duality to relate $\mathcal{N} = 4$ super Yang-Mills theory on a half-space to Chern-Simons theory in the boundary: we saw that it was most preferable to do this on a half-space $\mathbb{R}^3 \times \mathbb{R}_-$. This gave a new way to compute the Jones polynomial by counting solutions in an elliptic boundary value in 4 dimensions. Lifting the theory 1 dimension higher then allowed us to reinterpret the Jones polynomial as a trace in the space of supersymmetric vacua of the resulting 5-dimensional super Yang-Mills theory. It was then argued that its space of vacua should be viewed as Khovanov homology.

We then ended with a discussion of the implications of the latter application: the algebraic structure of Chern-Simons theory. A discussion of the role of modularity in Chern-Simons theory was given and the M-theory setting of the duality between Chern-Simons theory and $\mathcal{N} = 4$ SYM was highlighted. In the latter case, we reviewed a recent conjecture that the Poincaré polynomial of Khovanov homology can be explicitly computed using refined Chern-Simons theory, in a setting that is closely related to the M-theory setting of [29].



The main theme of this thesis has been the relation between geometry and physics and the key role that various dualities play in understanding the structure of models that embody this connection. Starting with 3-dimensional Chern-Simons, we saw that it can be totally solved by its relation with 2-dimensional conformal field theory through the Wess-Zumino-Witten model. Moreover, Chern-Simons theory can be equated to 4-dimensional supersymmetric gauge theory through the exotic cycles that we discussed. Subsequently, this can be lifted to 5-dimensional gauge theory and the 6-dimensional worldvolume theory of M5-branes in M-theory. This establishes an intriguing cascade of dualities, which indicates the richness of this subject.

Open questions

Starting from the 1D-2D correspondence, we showed that the exotic duality leads to the A-model quantization of classical phase space. Quantization for topological non-trivial phase spaces is still an ambiguous and ill-understood subject. However, an application of better understanding would be the study of Chern-Simons theory with non-compact gauge group, which is conjectured to give a model of $2+1$D quantum gravity and should lead to new knot invariants.

Going up in dimension, showing the gauge theory proposal for Khovanov homology is still an open question. We already indicated what should be done: one should deduce the algebraic rules of Khovanov homology from the gauge theory description. Furthermore, we saw that there is a new way to compute the Jones polynomial: by counting classical ground states in twisted $\mathcal{N}=4$ SYM after S-duality. This leads to a new link to statistical physics: the Gaudin model of XXX spin chains. This link is yet to be developed further.

It would be interesting to see if knot invariants for other compact gauge groups, such as Kauffman's polynomial for $G=SO(N)$ can be given an analogous gauge theory description using exotic integration cycles. In any case, one can construct $SO(N)$ gauge theory (more generally, any simply-laced group, with ADE root system) on a stack of D-branes on orientifolds, so that case superficially seems to admit a straightforward generalization of the relevant construction.

Discussing modularity in Chern-Simons theory, we also mentioned that a better insight in Chern-Simons theory with non-compact gauge group can be obtained by studying M5-branes on $M\times S^3$, which after appropriately compactifying leads to $SL(2)$ Chern-Simons theory and $\mathcal{N}=2$ SYM. Understanding this relation and generalizations is related to the AGT duality, the notion of geometric engineering and could lead to a more complete picture of the internal structure of these gauge theories.

In chapter 8 we discussed how to find exotic integration cycles for Chern-Simons theory on M, where we argued that adding Wilson loops on a knot K does not change the convergence of the path integral on an exotic integration cycle, as the Wilson loop is linear in the gauge fields in the exponential, while the Chern-Simons action is cubic. This is related to the scaling limit in which k becomes large and the representation of the Wilson loops remains fixed, that is, the highest weight of the representation (the 'electric' charge) that the Wilson loop sits in, is kept fixed. In this picture, we are describing Chern-Simons at weak coupling. However, one can look at the case where the charge n of the Wilson loop and the Chern-Simons coupling k are sent to ∞ while $\frac{n}{k}$ is kept fixed. This leads to the knot cobordisms of Khovanov homology and allows to complexify the Jones polynomial in terms of exotic cycles that come from critical orbits that include monodromy around K. This picture is related to the Volume conjecture in knot theory, which is a statement about the asymptotic behavior of the N-colored Jones polynomial in the limit of $N\to\infty$. This statement has been partially understood in [28].

Finally, A-branes and B-branes are conjectured to be related through homological mirror symmetry, the mathematical counterpart to physical mirror symmetry. The latter can be regarded as generalized T-duality when the space-time allows torus fibers. In general, mirror symmetry is unproven, but is strongly related to the geometric Langlands program, which revolves around a set of deep conjectures in algebraic geometry and related areas, such as category theory. Especially, it gives a geometric reformulation of the Langlands program, which spans a wide array of conjectures in number theory. Gauge theory, branes and S-duality play a big role in geometric Langlands, as worked out in [17].

This list is of course far from comprehensive, but already indicates that much more remains to be discovered in the intersection of mathematics and physics.

A

Mathematical background

In this appendix, we discuss some relevant mathematical concepts. Further references for homology and cohomology theory, fiber bundles and characteristic classes are [48, 49], which are not discussed here.

A.1 A note on equivariant cohomology

Starting from a more abstract point of view, a possible motivation to study symmetry phenomena on manifolds is the following observation. Suppose we have a continuous vector field X with isolated zeroes on a compact oriented 2-dimensional manifold Σ, then by the *Gauss-Bonnet theorem* and the Poincaré-Hopf theorem for the index of any smooth vector field X with isolated zeroes, we have

$$\frac{1}{2\pi}\int_{\Sigma} KdA = \chi(\Sigma) = \sum_{p\in\text{Zeroes}(X)} \text{index}_X(p). \tag{A.1.1}$$

This is a rather remarkable result: the left-hand-side computes topological information, whereas the right-hand-side is a discrete sum. This raises the question whether or not in general there is something to gain from fixed points ($x \in M$ such that $\forall g \in G : g \cdot x = x \implies h = e$) of a symmetry group G acting on M. This behavior already hints at the connection with supersymmetric localization, discussed in ??.

So consider a group G acting on a smooth manifold M. If G has no fixed points, then the quotient M/G is again a smooth manifold and we can *define* the equivariant cohomology of M to be just the cohomology of M/G. However, the simple example of rotations around the z-axis on the 2-sphere shows that this goes wrong in the most simple cases: in general M/G has singular points at fixed points of G. How do we define the equivariant cohomology in this case? The answer is that we should use a space of equivalent homotopy type (has isomorphic homotopy groups) that is canonically constructible from our initial data G and M, on which G acts freely. For this, we will define the space $M \times EG$, where G acts freely on the *contractible* space EG. Note that we need EG to be contractible in order to preserve the homotopy type in going from M to $M \times EG$. Then the equivariant cohomology of M is defined by

$$H^\bullet_G(M) = H^\bullet((M \times EG)/G) \equiv H^\bullet(M_G), \tag{A.1.2}$$

where we modded out by the diagonal action of G on $M \times EG$. So how do we construct EG? The canonical way to define EG is by considering the *universal bundle* $EG \longrightarrow BG$, which has the property that *every principal G-bundle* $E \longrightarrow M$ is a pullback. More precisely, this means that for every principal G-bundle $E \longrightarrow M$, there is a *classifying map* $f : M \longrightarrow BG$ such that the bundle $E \longrightarrow M$ is isomorphic to $f^\bullet(EG \longrightarrow BG)$. BG is called the *classifying space* of G and one can show that it is uniquely determined up to homotopy.

The most important example is that of the simplest case $M = \{\text{pt}\}$, the worst-case scenario, as G can only act trivially on M. Then we see easily that $\text{pt}_G = (\text{pt} \times EG)/G = EG/G = BG$, where the last equivalence holds since we only consider principal G-bundles here. Hence

$$H^\bullet_G(\text{pt}) = H^\bullet(BG). \tag{A.1.3}$$

Let us give an explicit example of a classifying space: let S^{2n+1} be the unit sphere in $\mathbb{C}^{n+1}$. Then we can a define an S^1-action on $\mathbb{C}^{n+1}$ by scalar multiplication, which is a free action; let $\exp(i\theta) \in S^1, \theta \in [0, 2\pi)$, then $(z^1, \ldots, z^n) \in \mathbb{C}^n \longmapsto \exp(i\theta)(z^1, \ldots, z^n) = (z^1, \ldots, z^n) \iff \theta = 0$. The quotient space is $\mathbb{C}P^n$ by definition. Since we want a contractible space, we need to consider the infinite union $S^\infty \equiv \cup_{n=0}^\infty S^{2n+1}$ and $\mathbb{C}P^\infty \equiv \cup_{n=0}^\infty \mathbb{C}P^n$, which are contractible*. Applying the statement from the previous paragraph, we get that the bundle

$$S^\infty \longrightarrow \mathbb{C}P^\infty \tag{A.1.4}$$

is a universal S^1-bundle and up to homotopy equivalence, $\mathbb{C}P^\infty$ is equal to BS^1.

Now we recall from algebraic topology that 'taking cohomology' $H^\bullet()$ is a *contravariant functor*. Hence, the constant map $M \longrightarrow \text{pt}$ induces a ring homomorphism $H^\bullet(\text{pt}) \longrightarrow H^\bullet(M)$. Hence, $H^\bullet(M)$ gets the structure of a module over the ring $H^\bullet(\text{pt})$: it is a vector space with coefficients in $H^\bullet(\text{pt})$, *for any M*.

Specializing to the previous S^1 action, the coefficient ring becomes

$$H^\bullet_{S^1}(\text{pt}, \mathbb{R}) = H^\bullet(\text{pt}_{S^1}, \mathbb{R}) = H^\bullet(BS^1, \mathbb{R}) = H^\bullet(\mathbb{C}P^\infty, \mathbb{R}) = \mathbb{R}[u] \tag{A.1.5}$$

where u is a generator of the cohomology $H^\bullet(\mathbb{C}P^\infty, \mathbb{R})$ of degree 2 associated to $\mathbb{C}P^1$ in the cell decomposition of $\mathbb{C}P^\infty = \mathbb{C}P^1 \cup \mathbb{C}P^2 \cup \mathbb{C}P^3 \cup \ldots$; it is just the Poincaré dual, a real 2-form, to $\mathbb{C}P^1$.

From this we learn that the coefficient ring in the case of the S^1-action consists of *polynomials in u*.

So far, our considerations were a bit abstract, but in fact, there is a more computational way of describing equivariant cohomology: namely using the language of equivariant differential forms: these are the forms ω that satisfy $L_g^\bullet \omega = \omega$ for all $g \in G$, where L_g denotes left multiplication. Such forms are determined by their value at the identity $e \in G$, so left-invariant forms constitue a finite-dimensional vector space $\Lambda^\bullet(\mathfrak{g}^\bullet)$: the exterior algebra generated by $\mathfrak{g}^\bullet$, the dual to $\mathfrak{g}$. This space inherits a differential operator d from $\Omega^\bullet(G)$, upon which we can interpret it as a differential complex $\Omega^\bullet(\mathfrak{g}^\bullet)$. For compact* connected G, this descends to an isomorphism on cohomology: $H^\bullet(\Omega^\bullet(\mathfrak{g}^\bullet)) \cong H^\bullet(\Omega^\bullet(G)) = H^\bullet(G)$. It is clear by the connectedness of G and homotopy invariance of de Rham cohomology, that L_g acts trivially in $H^\bullet(G)$: every class in $H^\bullet(G)$ is left-invariant.

What do we learn from this? The cohomology of a compact connected Lie group G is exactly determined by an infinitesimal description, namely it is determined in terms of the *structure constants* of $\mathfrak{g}$. Explicitly, if we have a basis $\{e_i\}$ for $\mathfrak{g}$ with $[e_i, e_j] = c_{ij}{}^k e_k$ then the dual base $\{\theta^i\}$ satisfying $\theta^i(e_j) = \delta^i_j$ generates $\Omega^\bullet(\mathfrak{g})$ and by the identity

$$d\theta^i + \frac{1}{2} c^i{}_{jk} \theta^j \theta^k = 0 \tag{A.1.6}$$

describes it completely.

So let us describe equivariant cohomology from this infinitesimal point of view. Let $X \in \mathfrak{g}$, then we have the Lie derivative in the direction of X, which satisfies

$$\mathcal{L}_X = d\iota_X + \iota_X d = (d + \iota_X)^2 \equiv \mathcal{D}_0^2. \tag{A.1.7}$$

Any $\mathfrak{g}$ invariant form will be annihilated by $\mathcal{L}_X$ and it is only such forms that we want to consider. We will call such forms *basic forms*.

Consider the space $W(\mathfrak{g}) = \Lambda^\bullet(\mathfrak{g}^\bullet) \otimes \text{Sym}(\mathfrak{g}^\bullet)$. Here $\text{Sym}(\mathfrak{g}^\bullet)$ is the symmetric tensor algebra on $\mathfrak{g}^\bullet$ that contains symmetric tensors like $\frac{1}{2}(v \otimes w + w \otimes v)$. It is crucial observation that this space should be regarded as the space of polynomials on $\mathfrak{g}^\bullet$: they are expressions that provide a map $\mathfrak{g} \longrightarrow \mathbb{R}$ that

*This is implied by the vanishing of all homotopy groups of S^∞. One can argue for this as follows: note that S^k is compact, so the image of any continuous map $S^k \longrightarrow S^\infty$ will be contained in some S^{2n+1} for some n. If n is large enough, any such map will be homotopic to the identity. Since we constructed S^∞ as a union of spheres $S^{2n+!}$, all homotopy groups vanish, hence it is contractible. This descends automatically to the quotient $\mathbb{C}P^n$

*Compactness is crucial, since in that case we can average forms over G, which projects $\Omega^\bullet(G)$ to $\Omega^\bullet(\mathfrak{g}^\bullet)$

moreover consists of elements with a discrete grading.

The generators of $W(\mathfrak{g})$ are then given by θ^i, the generators of $\Lambda^\bullet(\mathfrak{g}^\bullet)$ and $u^i = 1 \otimes \theta^i$, the generators of $\mathrm{Sym}(\mathfrak{g}^\bullet)$. This space can be interpreted as a complex when we equip it with the exterior derivative d_W defined by

$$d_W \theta^i = d\theta_i + \frac{1}{2} c^i{}_{jk} \theta^j \theta^k = u^i, \qquad d_W u^i = du^i + c^i{}_{jk} \theta^j u^k = 0. \tag{A.1.8}$$

So what are the invariant forms that we want to consider here? Since we have

$$\iota_{e_i} \theta^j = \delta_i^j, \qquad \iota_{e_i} u^j = 0, \tag{A.1.9}$$

we see immediately that in fact the invariant forms that we want only contain u's, that is, the invariant forms are polynomials in u.

Consider now the following complex

$$\Omega_G^\bullet(M) = ((W(\mathfrak{g}) \otimes \Omega^\bullet(M))^G, \mathcal{D}_0), \tag{A.1.10}$$

which is called the *twisted de Rham complex*. Then the key result by Cartan is that

$$H_G^\bullet(M, \mathbb{C}) = H^\bullet(\Omega_G^\bullet(M), \mathcal{D}_0). \tag{A.1.11}$$

This description of equivariant cohomology is called the *Cartan model*. It can be shown that the constraint $\iota_X \omega = 0$ eliminates all terms with θ, and one can also just use

$$\Omega_G^\bullet(M) = ((\mathrm{Sym}(\mathfrak{g}^\bullet) \otimes \Omega^\bullet(M))^G, \mathcal{D}_0), \tag{A.1.12}$$

where the differential $\mathcal{D}_0$ is given by

$$\mathcal{D}_0 u^i = 0, \qquad \mathcal{D}_0 \omega = d\omega - \iota_{e_i} \omega u_i. \tag{A.1.13}$$

Example

Consider $G = S^1$, then $W(\mathfrak{g}) = \mathbb{R}[\theta, u]$ is the exterior algebra with a single generator θ and a polynomial algebra in the element u of degree 2. Then an element in $\Omega_G^\bullet(M)$ will be of the form

$$\omega = \omega_0 + \theta \omega_1, \tag{A.1.14}$$

where the ω_i are polynomials in u with coefficients in $\Omega^\bullet(M)$. Now $\mathfrak{g}$ will have 1 basis element, denoted by X, so that $\iota_X \theta = 1$. Then ω will be basic iff

$$\iota_X \omega = 0, \qquad \mathcal{L}_X \omega = 0. \tag{A.1.15}$$

The first condition is

$$\iota_X(\omega_0 + \theta\omega_1) = \iota_X \omega_0 + (\iota_X \theta)\omega_1 - \theta \iota_X \omega_1 = 0, \tag{A.1.16}$$

so separating contributions of different degrees we have

$$\iota_X \omega_0 + (\iota_X \theta)\omega_1 = 0, \qquad \theta \iota_X \omega_1 = 0. \tag{A.1.17}$$

Note that $\iota_X^2 = 0$, so the first equation implies the second, and we are left with the characterization $\omega_1 = -\iota_X \omega_0$. The subspace generated by such forms is called the *basic subcomplex*.

Consider now the general case: a closed form in $\Omega_{S^1}^\bullet(M)$ in the Cartan model is represented by a polynomial in u, whose coefficients are X-invariant forms $\omega_i \in \Omega_X(M)$, denoted by

$$\omega = \omega_0 + u\omega_1 + \ldots + u^n \omega_n \tag{A.1.18}$$

where $n = \dim M$, satisfying $\mathcal{D}_0 \omega = 0$, which in terms of the coefficients ω_i is

$$d\omega_0 = 0, \qquad d\omega_1 = \iota_X \omega_0, \qquad d\omega_2 = \iota_X \omega_1, \qquad \ldots \qquad d\omega_n = \iota_X \omega_{n-1}. \tag{A.1.19}$$

Rotation on the 2-sphere.

As an example, let us return to the earlier case of rotations around the z-axis on S^2. The form

$$\omega = \frac{1}{4\pi}\frac{xdydz - ydxdz + zdxdy}{(x^2+y^2+z^2)^{3/2}} \tag{A.1.20}$$

is defined on $\mathbb{R}^3 - \{0\}$ and restricts to the normalized volume form on S^2 where $x^2+y^2+z^2=1$. The vector field that generates rotations around the z-axis is $X = 2\pi\left(x\partial_y - y\partial_x\right)$. Restricted to S^2, we use the constraint (??) to get on S^2

$$\begin{aligned}\iota_X\omega &= \frac{2\pi}{4\pi}\left(x^2dz + y^2dz - xzdx - yzdy\right) = \frac{1}{2}\left((x^2+y^2)dz - z(xdx+ydy)\right)\\ &= \frac{1}{2}\left((x^2+y^2)dz\right) + \frac{1}{2}z^2dz - \frac{1}{2}\left(z(xdx+ydy) - z^2dz\right) = \frac{1}{2}\left(dz - z(xdx+ydy+zdz)\right).\end{aligned}$$

But the second term $xdx+ydy+zdz = 0$ on S^2, this follows by taking the exterior derivative of the defining equation for the 2-sphere:

$$d\left(x_1^2+y^2+z^2\right) = d(1) = 0 \implies xdx+ydy+zdz = 0. \tag{A.1.21}$$

So we have $\iota_X\omega = \frac{dz}{2}$, on S^2. We see that the Hamiltonian is given by $\frac{z}{2}$ and an equivariant class in $H^2_{S^1}(S^2)$ is given by $\omega + \frac{z}{2}$u. Note that u is a generator for the symmetric tensor algebra $\mathrm{Sym}((T_eS^1)^*) = \mathrm{Sym}(\mathbb{R}^*) = \mathrm{Sym}(\mathbb{R})$.

A.2 The moment map

Looking at the case $n=1$ in particular, an equivariant class in $H^2_{S^1}(M)$ can be written as

$$\omega = \omega' + u\cdot H \tag{A.2.1}$$

where ω' is an invariant form on M and H is a function such that

$$\iota_X\omega = dH. \tag{A.2.2}$$

This equation leads to a *moment map* μ. Suppose M is a symplectic manifold with some symplectic form Ω. In this case, equation (A.2.2) can be inverted: given any $H\in\Omega^0(M)$, there is a unique vector field X_H such that $\iota_{X_H}\Omega = dH$. X_H is the *Hamiltonian flow* generated by H. Conversely, if we have an S^1-action generated by X that preserves the symplectic form, $\mathcal{L}_X\Omega = 0$, then this S^1-action is said to admit a *moment map* precisely if there is a function H satisfying (A.2.2). This extends to the case of general compact connected Lie groups G. It remains to explain what the moment map is.

In general, if a G-action preserves Ω, the function H is determined by a moment map $\mu : M \longrightarrow \mathfrak{g}^*$, such that for all vectors $\xi\in\mathfrak{g}$ and $x\in M$ we have $H_\xi(x) = \langle\mu(x),\xi\rangle$. The defining equation for the moment map then becomes

$$\iota_{X_{H_\xi}}\Omega = dH_\xi = d\langle\mu,\xi\rangle, \qquad \forall\xi\in\mathfrak{g}. \tag{A.2.3}$$

When this condition holds, we say that G-action is *Hamiltonian*. Note that from the defining equation for the moment map, it is clear that the space $\mu^{-1}(0)$ is a G-invariant subspace of M. If 0 is a regular value of μ, it follows that $\mu^{-1}(0)$ is a manifold and if G acts freely and properly on it, $\mu^{-1}(0)/G$ is also a manifold. We mention that when we start out with a Hamiltonian H and ξ is the vectorfield X_H generated by H, we have the tautological notation $H_{X_H} = H$.

Rotations in $\mathbb{R}^3$. Consider the phase space of classical mechanics on $\mathbb{R}^3$, which is equal to $T^*\mathbb{R}^3$. With coordinates $(x_1,x_2,x_3,y_1,y_2,y_3)$ on $T^*\mathbb{R}^3$ we have the symplectic form $\omega = dx_i\wedge dy_i, i=1,2,3$.

Suppose $G = SO(3)$. Then an element in $\mathfrak{g}$ generates the vectorfield $X = a_{ij}x_i\frac{\partial}{\partial y_i}$ where $a_{ij} = -a_{ji}$. The Lie-derivative is

$$\mathcal{L}_X dx_j = d(\mathcal{L}_X x_j) = d(a_{ij}x_i) = a_{ij}dx_i. \tag{A.2.4}$$

We want that $\mathcal{L}_X\omega = 0$, so we deduce that the right vectorfield X_ξ is given by

$$X_\xi = a_{ij}\left(x_i\frac{\partial}{\partial x_j} + y_i\frac{\partial}{\partial y_j}\right) \tag{A.2.5}$$

since

$$\mathcal{L}_{X_\xi}\omega = \mathcal{L}_{X_\xi}(dx_i \wedge dy_i) = (\mathcal{L}_{X_\xi}dx_i) \wedge dy_i + dx_i \wedge (\mathcal{L}_{X_\xi}dy_i) \tag{A.2.6}$$

$$= a_{ki}dx_k \wedge dy_i + dx_i \wedge a_{ki}dy_k = (a_{ki} + a_{ik})dx_k \wedge dy_i = 0. \tag{A.2.7}$$

From this, we compute

$$\iota_{X_\xi}\omega = a_{ij}\left(x_i dy_j - y_i dx_j\right) = a_{ij}\left(x_i dy_j + y_j dx_i\right) = a_{ij}d(x_i y_j) = \sum_{i<j} a_{ij}d(x_i y_j - y_i x_j), \tag{A.2.8}$$

from which we see that the moment map is given by

$$\mu(x_i, y_i)^i = \epsilon^{ijk}x_j y_k. \tag{A.2.9}$$

This is familiar: the moment is equal to the angular momentum and it is conserved because $\mathbb{R}^3$ admits rotational symmetry under $G = SO(3)$.

A.3 Symplectic and complex geometry

Symplectic geometry

Given M an $2n$-manifold, a symplectic form is a closed, non-degenerate 2-form ω. Non-degeneracy means that $\forall p \in M$ if $\forall Y \in T_pM : \omega(X,Y) = 0 \implies X = 0$. Since antisymmetric forms are not invertible in odd dimension, M should be even-dimensional. One can always find local coordinates, called the Darboux basis, such that ω is in the standard form

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \tag{A.3.1}$$

Given a vector subspace $W \subset V$, one can define the subspace perpendicular to W in V with respect to ω, namely

$$W^\perp = \{v \in V \mid \omega(w,v) = 0, \ \forall w \in W\}. \tag{A.3.2}$$

Note that $W^\perp \cap W$ does not necessarily vanish. This gives rise to the following nomenclature: if $W \subset W^\perp$, W is isotropic. If $W^\perp \subset W$, W is co-isotropic. If $W = W^\perp$, W is Lagrangian.

Complex geometry

An *almost complex structure* J on M^n is a automorphism of the tangent space $J : T_pM \longrightarrow T_pM$ that squares to -1: $J^2 = -1$. Note that *locally*, we always have a canonical form for

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and that we can always patch together such pointwise defined almost complex structures. Also note that necessarily we need the dimension of M to be even. We call M^n equipped with an almost complex structure J an *almost complex manifold.* Especially, almost complex manifolds are always orientable.

An almost complex structure J on M^{2n} enables us to at least *pointwise* define a notion of complex coordinates: note that the eigenvalues of J are $\pm i$. If $(x_i, y_i)_{i=1...n}$ are local coordinates on M^{2n}, we may pairwise define complex coordinates as $z_j = x_j + iy_j$. Multiplication by i on z, $z \mapsto iz$ then translates into $(x_1, y_1, \ldots, x_n, y_n)^t \mapsto J(x_1, y_1, \ldots, x_n, y_n)^t$. Note that J is an isometry with respect to the metric of M: $g(JX, JY) = g(X, Y)$.

Note that an almost complex structure *does not automatically* allow one to define local coordinate patches with holomorphic z^i and antiholomorphic $\bar{z}^i$ coordinates around every point $p \in M$. If M has holomorphic coordinate charts around every point, they will patch together into a holomorphic atlas that induces the almost complex structure J, J is then said to be *integrable* and M^n then is a *complex manifold.* We note that for surfaces, being an almost complex manifold is equivalent to being a complex manifold.

By the *Newlander-Nirenberg theorem*, J is integrable iff the Nijenhuis-tensor N_J of J vanishes on every pair of vectors X, Y:

$$N_J(X, Y) = [X, Y] + J[X, JY] + J[JX, Y] - [JX, JY] = 0, \tag{A.3.3}$$

which in index notation reads

$$N_{ij}^k = J_i^l \left(\partial_l J_j^k - \partial_j J_l^k\right) - J_j^l \left(\partial_l J_i^k - \partial_i J_l^k\right). \tag{A.3.4}$$

Moreover, we can introduce, analogous to the real case, basis elements for the tangent and cotangent space. Elements of the latter are then called (p, q)-form, if they contain p holomorphic forms and q anti-holomorphic forms.

Kähler structure

On a complex manifold M we can put at every point a *hermitian metric*: a positive-definite inner product $g : TM \otimes \overline{TM} \longrightarrow \mathbb{C}$. In index notation, the non-zero are exactly $g_{i\bar{j}}$ and we write $g = g_{i\bar{j}} dz^i d\bar{z}^{\bar{j}}$. This makes $g_{i\bar{j}}$ into a hermitian matrix. Using the metric, we can define the associated *Kähler form*: a $(1, 1)$-form ω that is locally given by $\omega = i g_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}}$. More intrinsically, we have

$$\omega(X, Y) = g(JX, Y).$$

This condition says that (g, ω, J) is a *compatible triple.* Note that for holomorphic vectors X, $JX = iX$. We then say that the metric is *Kähler* if $d\omega = \partial\omega + \bar{\partial}\omega = 0$, and we say that M is Kähler. Closedness of the Kähler form implies that

$$\partial_k g_{i\bar{j}} = \partial_i g_{k\bar{j}}, \qquad \partial_{\bar{k}} g_{i\bar{j}} = \partial_{\bar{j}} g_{i\bar{k}}. \tag{A.3.5}$$

From this we learn that locally, the metric can be written as $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \Phi$, where Φ is the Kähler potential. It follows straightforwardly that the only nonzero entries of the Levi-Civita connection are

$$\Gamma^i_{jk} = g^{i\bar{l}} \frac{\partial}{\partial z^k} g_{j\bar{l}}, \qquad \Gamma^{\bar{i}}_{\bar{j}\bar{k}} = g^{l\bar{i}} \frac{\partial}{\partial \bar{z}^k} g_{l\bar{j}} \tag{A.3.6}$$

and the curvature tensor has nonvanishing components

$$R_{i\bar{j}k\bar{l}} = -g_{m\bar{j}} \frac{\partial}{\partial \bar{z}^l} \Gamma^m_{ik}, \qquad R_{i\bar{j}k\bar{l}} = -R_{\bar{j}ik\bar{l}} = R_{\bar{j}i\bar{l}k} = R_{k\bar{j}i\bar{l}} \tag{A.3.7}$$

We see that on Kähler manifolds the Levi-Civita connection has pure indices: non-zero elements have only holomorphic of anti-holomorphic indices, the consequence is that holomorphic vectors remain holomorphic under parallel transport. Therefore, on a n complex dimensional complex manifold, the holonomy on a Kähler manifold sits in $U(n)$. The Laplacian satisfies $\Delta = d\delta + \delta d = 2\Delta_\partial = 2\Delta_{\bar{\partial}}$ and so harmonic forms are harmonic for all operators.

The simplest example is the complex plane $\mathbb{C}^n$, whose Kähler potential is given by the norm $z_i\bar{z}^i$. As another example, on complex projective space $\mathbb{CP}^1$, the Kähler potential is given by $\log(z\bar{z})$. A special and more intricate example is one of the two unique complex Kähler surfaces: the *K3 surface*, the other one being the complex torus T^4.

Calabi-Yau manifolds

Calabi-Yau manifolds are n-complex dimensional complex manifold that have holonomy in $SU(n)$. There are many equivalent ways to characterize them: they are Ricci-flat, have a non-vanishing maximal holomorphic $(n,0)$-form, have vanishing first Chern-class and are Kähler since $SU(n) \subset U(n)$. By Yau's theorem, a Kähler manifold with vanishing first real Chern class always admits a Ricci-flat Kähler metric. In 1 complex dimension, there is only the torus. In 2 complex dimensions, the only compact simply-connected Calabi-Yau manifolds are the $K3$-surfaces. There are also many non-compact examples. In complex dimension 3, the number of Calabi-Yau manifolds is bewildering large, which underpins the string theory landscape problem, since any Calabi-Yau 3-fold furnishes a possible superstring compactification. A concrete example is the quintic 3-fold, which is described as the codimension 1 hypersurface $\sum_{i=1}^{5}(X^i)^5 = 0$ where the X^i are homogeneous coordinates on $\mathbb{C}P^4$.

Hyperkähler structure

By definition, a hyperkähler manifold M has dimension $4k$ with k a positive integer, whose holonomy is contained in $Sp(k)$: the symplectic group $Sp(k)$ consists of matrices that preserve the hermitian metric $\langle x, y\rangle = \bar{x}_i y^i$ on $\mathbb{C}^k$. Such manifolds have the following distinguishing feature: they posses an S^2-space of complex structures. This means that there are three complex structures I, J, K on TM, that possess a quaternionic structure, that is $IJ = -JI, IJK = -1$. If $(a, b, c) \in S^2$, it is easy to check that $aI + bJ + cK$ then again squares to -1, and so is a complex structure (integrability follows from integrability from I, J, K). By definition, since $Sp(k) \subset SU(4k)$, such manifolds are Calabi-Yau.

Examples of these spaces: by Kodaira's classification, for $k = 1$ there is only T^4 or the K3-surface. For higher k, a notable example are the *asymptotically locally Euclidean* (ALE) spaces. Since $SU(2) \cong Sp(1)$, any Calabi-Yau surface is hyperkähler and vice versa.

A.4 Category theory and topological field theory

So far we have seen a description of topological field theory using the tools of (supersymmetric) quantum field theory, which unavoidably includes the use of the path integral formulation. However, only in a few special cases the path integral integration measure (e.g. $\mathcal{D}\phi$) can be given a rigorous mathematical description. To circumvent this problem one can try to give a purely *axiomatic* description of quantum field theory. This amounts to giving a *categorical* description in which a given quantum field theory should be viewed as an *abstract map* with *functorial* properties. Topological field theory is most easily amenable to such an abstract description, due to its virtue of being insensitive to the local properties of the spacetime it is defined on, making it possible to axiomatize its properties under surgery, cutting and gluing procedures (it is less clear to what ordinary QFT can be understood in this fashion). This abstract, formal, characterization of field theory will be useful when discussing topological Chern-Simons theory and mirror symmetry.

The main framework needed is *category theory* which provides a general way of looking at structure in mathematics.* A category C contains a collection ob(C) of *objects*, a collection hom(C) of arrows between objects, which we call *morphisms*, and the binary operation $\circ$ which is the *composition of morphisms*. The operation $\circ$ satisfies a number of elementary properties: the existence of an identity and associativity. An example is the category Sets which contains all possible sets as its objects. For these,

*The use of category theory is motivated for instance by realizing that the notion of the *set that contains all possible sets* is an intrinsically ill-defined concept.

the morphisms are just maps between sets. We can also consider all sets with the structure of a differentiable or smooth manifold, to get the category of smooth manifolds Diff. Then the morphisms are diffeomorphisms between manifolds.

Since a category can be seen as an object in itself, we can look for a map between categories that preserves the structure contained within them: such maps should preserve associativity and the identity. We call such maps *functors*, which come in two flavors: covariant functors preserve the direction of arrows, while contravariant functors reverse the direction of arrows.

Going back to the category Diff, a particularly nice example of a functor is the tangent functor $\mathcal{T}$. For suppose we have a diffeomorphism $f : M \longrightarrow N$ between smooth manifolds, $\mathcal{T}$ acts as

$$\{f : M \longrightarrow N\} \overset{\mathcal{T}}{\Longrightarrow} \{Df : TM \longrightarrow TN\}\,. \tag{A.4.1}$$

In this way, we see that $\mathcal{T}$ is a covariant functor between the categories Diff and Vect, the category of vector spaces.*

Cutting and pasting

In this abstract sense, an n-dimensional topological field theory should be seen as a *functor* $\mathbf{TFT}_n$. What does this have to do with physics? Any field theory is defined on a manifold M of dimension n which we think of as space-time. In general, if M has a nonempty boundary which might consist of several disjoint components, we may view it as a *cobordism* between two surfaces Σ_1 and Σ_2. We may view a cobordism between two d-dimensional manifolds Σ_1, Σ_2 by definition as some $d+1$-dimensional manifold whose boundary ∂M consists of the disjoint union $\Sigma_1 \cup \Sigma_2$. If M has no boundary, we can think of M as a cobordism between two empty manifolds, two empty sets. Moreover, we can glue two manifolds M, N with boundaries $\partial M = \Sigma_1 \cup \Sigma_2$ and $\partial N = \Sigma_1' \cup \Sigma_2'$ to each other if they have an identical boundary up to a diffeomorphism $y : \Sigma_2 \longrightarrow \Sigma_1'$. We can then view the glued product as a new cobordism $M' = M \cup_y N$ with boundary $\Sigma_1 \cup \Sigma_2'$. Note that 'gluing' corresponds to the composition $\circ$ of morphisms in Diff.

From now on, let M, N be two n-dimensional manifolds. In categorical language, we would like to think of the cobordism M as a morphism f between two objects that are assigned to its two codimension 1 boundaries Σ_1, Σ_2. Hence it is clear what $\mathbf{TFT}_n$ should do. $\mathbf{TFT}_n$ should be a functor that makes the above assignments and should be compatible with the gluing of corbordisms. Moreover, the most natural thing a topological field theory does is computing a partition function $Z(M)$, which as we have seen, should only depend on the topology of M. Hence, we want $\mathbf{TFT}_n$ to assign a number to a closed manifold.

We therefore postulate the following behavior: $\mathbf{TFT}_n$ assigns to the object '$n-1$-dimensional manifold $\Sigma^{(n-1)}$', another object 'a vector space $\mathcal{H}(\Sigma^{(n-1)})$', and assigns to a n-dimensional manifold with boundaries Σ_1, Σ_2 a morphism $f : \mathcal{H}(\Sigma_1) \longrightarrow \mathcal{H}(\Sigma_2)$. If Σ is empty, then $\mathbf{TFT}_n$ assigns to it the ring over which the vector spaces $\mathcal{H}$ are defined, which we just take to be $\mathbb{C}$ here.

Compatibility with gluing means the following. If $\mathbf{TFT}_n$ acts as

- $\mathbf{TFT}_n(\Sigma) = \mathcal{H}(\Sigma)$ where Σ is any $(n-1)$-dimensional boundary
- $\mathbf{TFT}_n(M) = f$ where $f : \mathcal{H}(\Sigma_1) \longrightarrow \mathcal{H}(\Sigma_2)$
- $\mathbf{TFT}_n(N) = g$ where $g : \mathcal{H}(\Sigma_1') \longrightarrow \mathcal{H}(\Sigma_2')$

and we can glue M to N; i.e. we have a diffeomorphism $y : \Sigma_2 \longrightarrow \Sigma_1'$, then we should have

$$\mathbf{TFT}_n(M \cup_y N) = \mathbf{TFT}_n(N) \circ' \mathbf{TFT}_n(M) = g \circ' f \tag{A.4.2}$$

where $\circ'$ denotes composition for the morphisms between the vector spaces $\mathcal{H}$: in this case, this is just composition of linear maps.

*Of course, one can continue in this fashion and consider morphisms between morphisms (bimorphisms), morphisms between bimorphisms and so on, leading to higher levels of structure such as n-categories and n-functors. A discussion of these topics is outside the scope of this text.

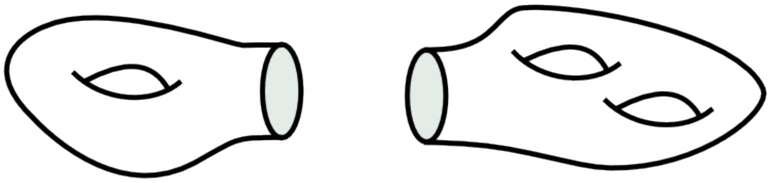


Figure 14: Surgery on manifolds.

So $\mathbf{TFT}_n$ assigns to a closed manifold M, which corresponds to a cobordism $\emptyset \longrightarrow \emptyset$, a morphism $f : \mathbb{C} \longrightarrow \mathbb{C}$. This is a linear map between two copies of $\mathbb{C}$, but that is just an element of $\mathbb{C}$. We interpret this number as the partition function $Z(M)$.

Cutting a closed manifold M in two pieces: $M = M_- \cup M_+$, splits the cobordism $\emptyset \longrightarrow \emptyset$ into two composite cobordisms $\emptyset \longrightarrow \Sigma_-$ and $\Sigma_+ \longrightarrow \emptyset$, where Σ_+ is Σ_- with the opposite orientation. Then we have

$$M_- \overset{\mathbf{TFT}_n}{\longrightarrow} f_- : \mathbb{C} \longrightarrow \mathcal{H}(\Sigma_-) \tag{A.4.3}$$

$$M_+ \overset{\mathbf{TFT}_n}{\longrightarrow} f_+ : \mathcal{H}(\Sigma_+) \longrightarrow \mathbb{C} \tag{A.4.4}$$

$$M_- \cup M_+ \overset{\mathbf{TFT}_n}{\longrightarrow} f_- \circ' f_+ : \mathbb{C} \longrightarrow \mathbb{C} \tag{A.4.5}$$

The map f_- is a linear map from $\mathbb{C}$ to a vector space $\mathcal{H}(\Sigma_-)$, hence we can view f_- as a vector itself in $\mathcal{H}(\Sigma_-)$. Likewise, f_+ is a linear functional, a covector in $\mathcal{H}^*(\Sigma_+)$. By gluing, we have that $f_+ \circ f_- : \mathbb{C} \longrightarrow \mathbb{C}$ is an element $\mathbb{C}$. We want to think of this element as $\langle f_+, f_- \rangle = f_+(f_-)$. Hence we see that $\mathbf{TFT}_n(M_-) = f_- = |M_-\rangle \in \mathcal{H}(\Sigma_-)$, so for consistency with gluing $\mathbf{TFT}_n$ should assign an element $f_+ = \langle M_+|$ in the dual Hilbert space $\mathcal{H}^*(\Sigma_+)$ which is isomorphic to $\mathcal{H}^*(\Sigma_-)$.
With the natural bilinear pairing $\mathcal{H}^*(\Sigma_-) \times \mathcal{H}(\Sigma_-) \longrightarrow \mathbb{C}$, we see that this assignment then is completely compatible with the composition of arrows and the behavior of $\mathbf{TFT}_n$, i.e. we have $\langle M_+|M_-\rangle = Z(M)$.

So why should $\mathbf{TFT}_n$ assign a Hilbert space to a boundary of M? Consider the path integral formulation of quantum mechanics where $x(t) \in \mathbb{R}^n$ is the trajectory of some quantum mechanical particle, where $t \in \mathbb{R}$. Then the path integral amplitude for a particle to travel between two points x_A, x_B is

$$\langle x_B | x_A \rangle = \int_{x(t_i)=x_A, x(t_f)=x_B} \mathcal{D}x(t) \exp\left(i \int dtL\right). \tag{A.4.6}$$

Especially, we can look at the special case that $t_i = -\infty, t_f = \infty$, whose amplitude we can compute as follows: calculate the amplitude for the particle moving from $x(-\infty) = x_A$ to some fixed $x(0) = x_0$, and calculate the amplitude for the particle moving from x_0 to $x(\infty) = x_B$. Then we obtain two amplitudes, which when multiplied and integrated over the position x_0 will give the full amplitude

$$\begin{aligned}\langle x_B | x_A \rangle &= \int dx_0 \langle x_B | x_0 \rangle \langle x_0 | x_A \rangle \\ &= \int_{\mathbb{R}^n} dx_0 \int_{x(-\infty)=x_A, x(0)=x_0} \mathcal{D}x(t) \exp\left(i \int dtL\right) \times \int_{x'(0)=x_0, x'(\infty)=x_B} \mathcal{D}x'(t) \exp\left(i \int dtL\right)\end{aligned}$$

Now we can interpret the position at $t = 0$ as a boundary: the amplitude $\langle x_B | x_0 \rangle$ as a function of the boundary condition $x(0) = x_0$ can be thought of as representing a state in a Hilbert space, which in this case is equal to $\mathbb{R}^2$. These arguments extend to field theory in a similar fashion: for instance, there the Hilbert space could be the space of all values a field can take at a certain space-time position.

B

Coupling to the $\mathcal{B}_{cc}$-brane

We start from the bosonic part of (6.1.13), which differs from the A-model action by a Bogomolny term. The bosonic kinetic term expands as:

$$
\begin{aligned}
Y_A \wedge *Y^A &= g_{AB}\left(d\xi^A \wedge *d\xi^B - d\xi^A \wedge *^2 I^B_C d\xi^C - *I^A_C d\xi^C \wedge *d\xi^B + *I^A_C d\xi^C \wedge *^2 I^B_D d\xi^D\right) \\
&= g_{AB}\left(d\xi^A \wedge *d\xi^B + d\xi^A \wedge I^B_C d\xi^C - I^A_C d\xi^C \wedge d\xi^B - *I^A_C d\xi^C \wedge I^B_D d\xi^D\right) \\
&= g_{AB}\left(\frac{\partial\xi^A}{\partial s}ds + \frac{\partial\xi^A}{\partial t}dt\right) \wedge \left(\frac{\partial\xi^B}{\partial s}dt - \frac{\partial\xi^B}{\partial t}ds\right) \\
&\quad g_{AB}\left(\frac{\partial\xi^A}{\partial s}ds + \frac{\partial\xi^A}{\partial t}dt\right) \wedge I^B_C\left(\frac{\partial\xi^C}{\partial s}ds + \frac{\partial\xi^C}{\partial t}dt\right) \\
&\quad - g_{AB}I^A_C\left(\frac{\partial\xi^C}{\partial s}ds + \frac{\partial\xi^C}{\partial t}dt\right) \wedge \left(\frac{\partial\xi^B}{\partial s}ds + \frac{\partial\xi^B}{\partial t}dt\right) \\
&\quad - g_{AB}I^A_C\left(\frac{\partial\xi^C}{\partial s}dt - \frac{\partial\xi^C}{\partial t}ds\right) \wedge I^B_D\left(\frac{\partial\xi^D}{\partial s}ds + \frac{\partial\xi^D}{\partial t}dt\right) \\
&= \left(g_{AB}\left(\frac{\partial\xi^A}{\partial s}\frac{\partial\xi^B}{\partial s} + \frac{\partial\xi^A}{\partial t}\frac{\partial\xi^B}{\partial t}\right) + 2\omega_{AB}\frac{\partial\xi^A}{\partial s}\frac{\partial\xi^B}{\partial t}\right) ds \wedge dt
\end{aligned}
$$

where we used $g_{AB}I^B_C I^A_D = -g_{CD}$ and $I^A_B = g^{AC}\omega_{CB}$. Inserting an overall paramter $2t$ as in chapter 3, the action then becomes

$$
I^{top}_A = 2t\left(\int_D dsdt g_{AB}\left(\frac{\partial\xi^A}{\partial s}\frac{\partial\xi^B}{\partial s} + \frac{\partial\xi^A}{\partial t}\frac{\partial\xi^B}{\partial t}\right) + 2\int_D dsdt\omega_{AB}\frac{\partial\xi^A}{\partial s}\frac{\partial\xi^B}{\partial t}\right) + \text{fermions}. \tag{B.0.1}
$$

We recognize the first term as kinetic term for the A-model in real coordinates, while the second term becomes with $\omega_{AB} = \partial_A c_B - \partial_B c_A$ (note that $\partial_A \equiv \frac{\partial}{\partial\xi^A}$):

$$
\begin{aligned}
\int_D dsdt\left(\partial_A c_B \frac{\partial\xi^A}{\partial s}\frac{\partial\xi^B}{\partial t} - \partial_B c_A \frac{\partial\xi^A}{\partial s}\frac{\partial\xi^B}{\partial t}\right) &= \int_D dsdt\left(\partial_A c_B \frac{\partial\xi^A}{\partial s}\frac{\partial\xi^B}{\partial t} - \partial_B c_A \frac{\partial\xi^A}{\partial s}\frac{\partial\xi^B}{\partial t}\right) \\
= \int_D dsdt\left(\frac{\partial c_B}{\partial s}\frac{\partial\xi^B}{\partial t} - \frac{\partial c_B}{\partial t}\frac{\partial\xi^B}{\partial s}\right) & \\
= \int_D dsdt\frac{\partial}{\partial s}\left(c_B\frac{\partial\xi^B}{\partial t}\right) - \frac{\partial}{\partial t}\left(c_B\frac{\partial\xi^B}{\partial s}\right) + \int_D dsdt c_B\left(\frac{\partial}{\partial s}\frac{\partial}{\partial t} - \frac{\partial}{\partial t}\frac{\partial}{\partial s}\right)\xi^B & \\
= \int_D c_B d\xi^B = h. &
\end{aligned}
$$

which is just the Morse function we used. Let us look at the second exponential: it reads

$$
\exp\left(i\oint \Lambda_A d\xi^A\right) = \exp\left(h + i\oint_{\partial D} b_A d\xi^A\right). \tag{B.0.2}
$$

Therefore, we see that the bosonic part of the integrand is

$$
\exp\left(-2t\left(\int_D dsdt g_{AB}\left(\frac{\partial\xi^A}{\partial s}\frac{\partial\xi^B}{\partial s} + \frac{\partial\xi^A}{\partial t}\frac{\partial\xi^B}{\partial t}\right) + i\oint_{\partial D} b_A d\xi^A\right)\right)\prod_{i=1}^{N} u_i(t_i)\mathcal{O}_V(0). \tag{B.0.3}
$$

The first term in the exponential is the standard σ-model kinetic term, but the second one comes from the term $\int p_i dq^i$ from the original path integral (6.1.1) we started out with. In the A-model, this factor should be interpreted as a boundary coupling of the A-model to the topological A-brane $\mathcal{B}_{cc}$.

C

Supersymmetry, geometry and vacua

In this chapter we show how Morse theory can be applied to analyze (supersymmetric) σ-models. Morse theory is a classic tool in differential geometry which can be used to study the topology of a manifold by studying scalar functions on M. An identification is that Morse theory flow lines correspond exactly to instantons in the quantum theory, that represent particles tunneling between classical vacua. Such instantons lift the energy of classical vacua, and this lifting is exactly captured by the Morse-Smale-Witten complex, which is defined on the critical points on M. This illustrates the intimate relation between geometry, analysis and physics. The material discussed here will mainly be used in chapter 8, where we use it to describe Khovanov homology. A reference for this material is [7].

C.1 Morse inequalities and Morse-Smale-Witten complex

Morse-Smale-Witten complex

The collection of all sets C_k of critical points of Morse index k form a (co)homological complex, with a boundary operator $\partial : C_k \longrightarrow C_{k-1}$ that counts how many downward flow lines there are from Morse index k to $k-1$. Alternatively, one can define a coboundary operator δ that counts upward flow lines from Morse index k to $k+1$. If the Morse-Smale condition holds, this coboundary operator exists and the cohomological complex is called the *Morse-Smale-Witten complex*. The construction of these operators is analogous to instanton calculations, which we shall discuss below. One can show that the dimension of the space of flow lines between two points exactly equals their difference $p-q$ in Morse index. Modding out by reparametrization invariance, the moduli space $\mathcal{M}(p,q)$ has dimension $p-q-1$, and has a natural compactification $\overline{\mathcal{M}(p,q)}$. For points differing 1 in Morse index, $\mathcal{M}(p,p-1)$ is a collection of signed points, and the coboundary operator acts precisely as $\partial p = \sum_{\mu(p)-1} \#\mathcal{M}(p,q)q$. The intuitive reason for nilpotency $\partial^2 = 0$ is that a downward flow from a point with Morse index p to Morse index $p-2$ always limits to a broken flow, a flow that interpolates between two consecutive points that pairwise differ 1 in Morse index. Hence the 0-dimensional moduli space of broken flows constitutes a boundary for the 1-dimensional space $\mathcal{M}(p,p-2)$, and also on the boundary of $\overline{\mathcal{M}(p,p-2)}$, which is compact and oriented. But in that case, all the signs have to cancel. Hence $\partial^2 = 0$.

Morse inequalities

An important result of Morse theory is the *weak Morse inequality*

$$b_k \leq N_k \tag{C.1.1}$$

between the Betti numbers $b_k = \dim H^k(M)$ and the number N_k of critical points of Morse index k. One also has the *strong Morse inequality*

$$\sum_{k=0}^{n} (N_k - b_k)\, t^k = (1+t) \sum_{k=0}^{n} Q_k t^k, \qquad Q_k \geq 0. \tag{C.1.2}$$

Inserting $t=-1$, we find that

$$\sum_{k=0}^{n} (-1)^k N_k = \sum_{k=0}^{n} (-1)^k b_k = \chi(M). \tag{C.1.3}$$

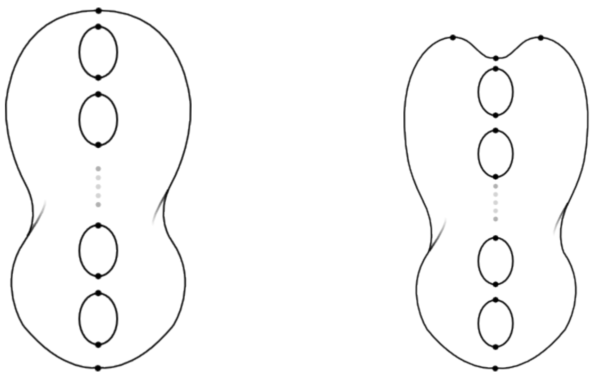
It turns out that the validity of these Morse inequalities is equivalent to the existence of the Witten complex, which we will describe in the coming section.

The height function on Σ_g

Let us consider an example in 2 dimensions. Take a Riemann surface Σ_g of genus g and position it such that all holes are aligned vertically. As our Morse function, we can take a linear height function (a gravitational potential) on Σ_g, which is clearly a Morse function on Σ_g. Then there are 2 critical points, 1 at the top and 1 at the bottom, and $2g$ saddlepoints. The Morse index of the upper critical points is 2 (it has 2 unstable directions) and that of the lower is 0. Every saddlepoint has Morse index 1. Then according to the identity (C.1.3) we have

$$\chi(M) = \sum_{k=0}^{2}(-1)^k N_k = 1 - 2g + 1 = 2 - 2g, \tag{C.1.4}$$

which is the familiar result. More importantly, consider a smooth deformation of Σ_g at the top, visualizable by pushing into the surface with your thumb and making a dent. Then we create one additional local minimum and one saddlepoint, which changes $\delta M_0 = 1, \delta M_1 = 1, \delta M_2 = 0$, such that the Euler characteristic is unchanged, as we expect from the topological nature of $\chi(M)$. It is straightforward to extend this to any smooth deformation.



(a) The $2g + 2$ critical points of the height function on Σ_g.

(b) A deformation of Σ_g and the extra critical points.

C.2 Supersymmetric ground states

Now follows our first application of Morse theory to the supersymmetric σ-model: we shall see how Morse theory precisely captures the space of supersymmetric ground states. The reference for this is [8].

Counting classical ground states

The identification (??) between geometrical objects (differential forms) and physical fields allows for a more physical proof of the Morse inequalities, as was first shown by Witten [8]. To analyze the behavior around a critical point, we rescale the superpotential $h \to \lambda h$. This is equivalent to a deformation of the exterior derivative and its adjoint:

$$d_\lambda = e^{-\lambda h} d e^{\lambda h} = d + \lambda \bar{\psi}^i \frac{\partial h}{\partial x^i}, \qquad d_\lambda^* = e^{\lambda h} d^* e^{-\lambda h} = d^* + \lambda \psi^i \frac{\partial h}{\partial x^i}. \tag{C.2.1}$$

It is immediate that $d_\lambda^2 = 0$, since we are conjugating by $e^{\lambda h}$.* Therefore the cohomology of d_λ is equivalent to that of d. Associated to this is the modified Betti number $b_q(M, \lambda)$:

$$b_q(M,\lambda) = \dim\left(\ker(d_\lambda d_\lambda^* + d_\lambda^* d_\lambda) \cap \Omega^p(M)\right) \tag{C.2.2}$$

*Explicitly $d_\lambda^2 = \left(e^{\lambda h} d e^{-\lambda h}\right)\left(e^{\lambda h} d e^{-\lambda h}\right) = e^{\lambda h} d^2 e^{-\lambda h} = 0$

which is the number of independent p-forms with respect to the modified differential d_λ. The modified Betti number will depend continuously on λ, but it values are discrete.
Hence, we can freely choose a value of λ to find $b_q(M) = b_q(M, 0)$ and can study the cohomology of d_λ, the vacua of the modified Hamiltonian $H_\lambda = \frac{1}{2}(d_\lambda d_\lambda^* + d_\lambda^* d_\lambda)$ in the limit $\lambda \to \infty$. It is straightforward to check that the modified Hamiltonian becomes:

$$2H_\lambda = (dd^* + d^*d) + \lambda^2 g^{ij} \frac{\partial h}{\partial x^i} \frac{\partial h}{\partial x^j} + \lambda[\overline{\psi}^i, \psi^j] D_i D_j h.$$

From the form of the Hamiltonian, we see that low-energy states lie near the critical points of the quadratic potential term. As $\lambda \to \infty$, these minima becomes localized at exactly the critical points.

When we expand around a critical point, we can always choose locally flat coordinates (a result from Riemannian geometry) such that $g_{ij} = \delta_{ij} + \mathcal{O}(x^2)$, so the Christoffel symbols Γ^{ijk} vanish to $\mathcal{O}(x)$ and $h(x) = h(0) + c_i x_i^2 + \mathcal{O}(x^3)$. Note that the number of negative c_i is exactly the Morse index $\mu(p)$ at the critical point. Hence we get the following expansion up to $\mathcal{O}(x^3)$:

$$2H_\lambda = \sum_i \left(-\frac{\partial^2}{\partial x_i^2} + \lambda^2 c_i^2 x_i^2 + \lambda c_i [\overline{\psi}^i, \psi^i] \right) \tag{C.2.3}$$

which is the Hamiltonian of a n-dimensional harmonic oscillator with a correction term

$$[\overline{\psi}^i, \psi^i] = \pm 1. \tag{C.2.4}$$

Here the commutator is $+1$ when $i \in \{i_1, \ldots, i_{\mu(p)}\}$ and -1 otherwise. Since the first two terms and the commutator-term commute, we can simultaneously diagonalize them. We note that the first two terms are the standard ones for harmonic oscillators, hence we can immediately deduce the spectrum of the Hamiltonian as

$$E_\lambda = \frac{1}{2}\lambda \sum_i (|c_i|(1 + 2N_i) + c_i n_i) + \mathcal{O}(x^0), \qquad n_i = \pm 1, \qquad N_i \in \mathbb{N}. \tag{C.2.5}$$

Now this will only give $E_\lambda = 0$ when $N_i = 0$ for all i and if $n_i = -\text{sign}\, c_i$. We have exactly $\mu(p)$ indices for which $n_i = +1$ from equation (C.2.4) and the defining property of the Morse index, hence the eigenfunction for a ground state with zero energy is a $\mu(p)$-form. We see that in the classical description, each critical point gives a suitable ground-state wave function Ψ_i^0 whose energy vanishes to order λ^2; we see that we have $N_{\mu(p)}$ such ground-states in $\Omega^{\mu(p)}(M)$. The exact form of Ψ_i^0 can be found by explicitly solving $Q\Psi_i^0 = \overline{Q}\Psi_i^0 = 0$, an example of which can be found in [7].

It turns out that this states remain in the classical spectrum up to all order in perturbation theory. A conceptual way to understand this is that perturbation theory is a local calculation: it is blind to the topology of M. The existence of a critical point and Ψ_i^0 however, *is* a topological issue. Therefore, perturbation theory cannot remove classical ground states, only non-local calculations of tunneling effects or *instantons*, can remove classical ground states from the ground state spectrum. Recall that classical ground states equal harmonic forms, whose number is coupled to the topology of M: this is compatible with our remarks above.

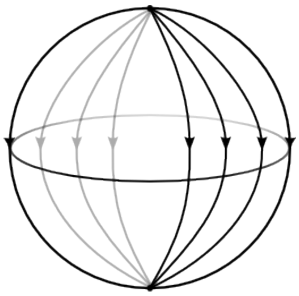
A physical proof of the weak Morse inequality (C.1.1) now easily follows: the number b_k of nonperturbative ground states (those states with exactly zero energy) is always smaller of equal the number N_k of ground states in first order perturbation theory. This simply implies the weak Morse inequality: $b_k \leq N_k$. This observation was first made by Witten [8].

So we conclude that if the Morse function h has non-degenerate critical points, classically every critical points contributes 1 state to the cohomology of M in the perturbative description.[††]

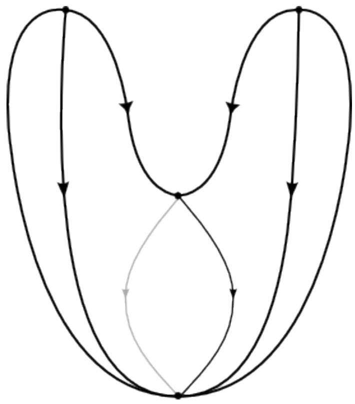
[††] A systematic treatment with degenerate critical points can be found in [8].

Example

Consider the height function h on the sphere Σ_0. Undeformed, h has two critical points, which correspond to two non-perturbative vacua, one 0-form and one 2-form, considering the Morse indices at the north and south pole. If we deform the sphere such that it gets two maxima and one saddle-point, h will have four critical points, giving 4 classical ground states: two 2-forms, one 1-form and one 0-form. However, the space of quantum ground states has to be a topological invariant: this indicates that some classical ground states will actually be lifted by instanton effects. Since in the undeformed case, both ground states are bosons, they cannot be lifted by instantons (also, they do not differ 1 in Morse index). Therefore, they are quantum ground states, and two states in the deformed case must be lifted.



(c) Morse flow on a sphere.



(d) Morse flow on a deformed sphere.

Instanton lifting of supersymmetric vacua in SQM

We want to illustrate by an explicit calculation how instantons can lift supersymmetric vacua in supersymmetric quantum mechanics, where we recall that supersymmetric vacua have $Q|0\rangle = \overline{Q}|0\rangle = 0$. We simplify the situation somewhat by taking a 1-dimensional target space, such that we have a 1-dimensional σ-model with $\Phi : \mathbb{R} \to \mathbb{R}$. Consider now the situation that V has two minima, with associated semi-classical ground states $|1\rangle, |2\rangle$, one bosonic and one fermionic state, whose exact (quantum) lowest-energy states are $|0_\pm\rangle$. Semi-classically, $|1\rangle, |2\rangle$ do not mix, since $\langle 1|H|2\rangle = 0$ by conservation of fermion number. However, quantum effects can lift this ground state degeneracy. Their common energy is given by

$$E_0 = \langle 0_+|H|0_+\rangle = \langle 0_-|H|0_-\rangle = \frac{1}{2}\langle 0_+|\left\{Q,\overline{Q}\right\}|0_+\rangle = \frac{1}{2}\langle 0_+|Q\overline{Q}|0_+\rangle = \sum_{k=\pm}\frac{1}{2}\langle 0_+|Q|k\rangle\langle k|\overline{Q}|0_+\rangle.$$

By conservation of fermion number again, this last sum reduces to $E_0 = \frac{1}{2}|\langle 0_-|\overline{Q}|0_+\rangle|^2$. Our goal is to calculate the supersymmetry breaking order parameter $\epsilon = \sqrt{2E_0} = \langle 0_-|\overline{Q}|0_+\rangle$. This matrix element is calculated in Euclidean time by the path integral

$$\begin{aligned}\int_{\phi(-\infty)=\phi_-}^{\phi(+\infty)=\phi_+}\mathcal{D}\phi(t)\mathcal{D}\psi\mathcal{D}\overline{\psi}\exp\left(-S/\hbar\right)\overline{Q}(t_0) &= \lim_{T\longrightarrow\infty}\langle 1|e^{-H(T/2+t_0)}\overline{Q}(t_0)e^{-H(T/2-t_0)}|2\rangle\\ &= e^{-E_0T/\hbar}\langle 1|0_+\rangle\langle 0_+|\overline{Q}|0_-\rangle\langle 0_-|2\rangle\\ &\approx \langle 0_+|\overline{Q}|0_-\rangle = \epsilon. \end{aligned} \tag{C.2.6}$$

where we used that $\langle x|i\rangle = \delta(x-x_i)$ and dropped the exponential since to lowest order $E_0 = 0$. In leading terms in $\hbar$ (so we may set $\frac{1}{2}\left[\overline{\psi},\psi\right] = \overline{\psi}\psi$, so the Euclidean version of the Lagrangian (??) is

$$L_E = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(h')^2 - \overline{\psi}\dot{\psi} - h''\overline{\psi}\psi. \tag{C.2.7}$$

Here we denoted derivatives with respect to the target space coordinate ϕ by primes. First we need to find the saddle-point contribution to the path integral. This is given by solving for a zero-energy classical solution. So the Euclidean Hamiltonian $\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}h'^2 = 0$, so we infer $\dot{\phi} = \pm h'$. Taking the $+$-solution, this implies by differentiation to time the classical equation of motion:

$$\ddot{\phi} - h''\dot{\phi} = \ddot{\phi} - h'h'' = 0, \qquad \phi(-\infty) = \phi_1, \qquad \phi(+\infty) = \phi_2. \tag{C.2.8}$$

We call the solution to this differential equation $\phi_{\text{cl}}(t)$. After expanding quadratically around the classical solution, one finds that the action becomes

$$\begin{aligned} S[\phi_{\text{cl}}(t)] + S_{\text{quad}} &= \Delta h + \frac{1}{2}\int dt\,\left(\delta\phi D_B \delta\phi + \delta\overline{\psi} D_F \delta\psi\right), \qquad \text{(C.2.9)}\\ D_B = -\partial_t^2 + h'''h' + (h'')^2 &= (-\partial_t - h'')(\partial_t - h'') = \overline{D_F} D_F, \qquad D_F = \partial_t - h''. \end{aligned}$$

Localization implies that we can now evaluate the path integral (C.2.6) exactly in the quadratic approximation, since the semi-classical approximation is exact. The presence of the $\overline{Q}$ insertion exactly kills a fermionic zero mode. Moreover, there is a bosonic zero mode. Just as in chapter 3, these could be calculated by an index formula, but in this case, the zero modes can be identified more easily. Namely

$$D_F \dot{\phi}_{\text{cl}} = (\partial_t - h'')\dot{\phi}_{\text{cl}} = \dot{h}' - h''h' = h''\dot{\phi}_{\text{cl}} - h''h' = 0. \tag{C.2.10}$$

Hence $\overline{\psi}$ has a zero mode $\delta\overline{\psi} = \overline{\eta}_0 \dot{\phi}_{\text{cl}}$, where $\overline{\eta}_0$ is a Grassmann constant. Since $D_B = D_F^2$, we see that D_B has a zero mode $\delta\phi = \epsilon_0\dot{\phi}_{\text{cl}}$ where ϵ_0 is any real constant. We note that $\overline{D_F}\delta\psi = (\partial_t + h'')\delta\psi = 0$ has no solution, hence ψ has no zero mode. The bosonic zero mode comes from time translation invariance and can be dealt with by integrating over all possible centers of time for the instanton. The path integral for the quadratic part reduces to the ratio

$$\epsilon = \exp\left(-\Delta h/\hbar\right)\frac{\det D_F}{\sqrt{\det D_B}} = \pm\exp\left(-\Delta h/\hbar\right), \tag{C.2.11}$$

where $\Delta h = h(x_+) - h(x_-)$ and the prefactor comes from the bosonic zero mode integration. The instanton therefore clearly lifts the classical ground state from zero energy, breaking supersymmetry.

Counting quantum ground states

Morse theory captures the corrections to the spectrum due to instantons, in the form of the Morse-Witten complex. This complex comes with a coboundary operator $\delta : X_p \longrightarrow X_{p+1}$, defined by upward flow. Here X_p is the set of points of Morse index p:

$$\delta|p\rangle = \sum_{q=p+1} n(p,q)|q\rangle. \tag{C.2.12}$$

Note that nilpotency of δ implies that a pair of ground states associated to critical points can only have their energies lifted by instantons when their Morse index differs by 1. The calculation of $n(p,q)$ is just the sum of all instanton contributions between $|p\rangle$ and $|q\rangle$, each instanton contributing a factor of ± 1 as in a rescaled version of (C.2.11).* To make this slightly more precise, consider the Landau-Ginzburg model given by (??). Using the *Bogomolny trick* we rewrite the bosonic part of the action as:

$$\begin{aligned} S_{phys} &= \frac{1}{2}\int_I dt\left(\lambda g_{ij}\dot{\phi}^i\dot{\phi}^j - \lambda g^{ij}\frac{\partial h}{\partial\phi^i}\frac{\partial h}{\partial\phi^j}\right) + \ldots = \int_I dt\left(\frac{1}{2}\lambda\left|\frac{\partial\phi^i}{\partial t} \mp g^{ij}\frac{\partial h}{\partial\phi^i}\right|^2 \pm \lambda\dot{h}\right) + \ldots \\ &= \int_I dt\left(\frac{1}{2}\lambda\left|\dot{\phi} \mp g^{ij}\frac{\partial h}{\partial\phi^i}\right|^2\right) \pm \lambda h|_{-\infty}^{+\infty} + \ldots = S_{top} \pm \lambda h|_{-\infty}^{+\infty}. \end{aligned} \tag{C.2.13}$$

*Geometrically, the number $n_\gamma = \pm 1$ is determined as follows. At every critical point A there is an associated state $|a\rangle$ which is a p-form $\omega_{p,A}$, if the Morse index $\mu(A) = p$. Then ω_p determines an orientation of the p-dimensional vector space V_A of negative eigenvectors of the Hessian of the Morse function f at A. Likewise, there is a natural orientation of V_B at B. Now we can consider a path γ that runs from A to B. If v is the tangent vector to γ at A, we denote by $V_A^\perp \subset V_A$ the subspace of V_A orthogonal to v at A. $V_A^\perp$ inherits an orientation from V_A, defined by the $p-1$-form obtained from interior multiplication with v: $i_v\omega_{p,A}$. Now from the Morse flow from A to B, we get a map from $V_A^\perp$ to V_B, which have both the same dimension $p-1$, moreover, we induce an orientation on V_B. The coefficient n_γ is then $+1$ if the induced orientation corresponds to the orientation that is determined by $\omega_{p,B}$ or -1 if it is opposite.

where the dots represent fermion terms. It is then immediately clear that as $\lambda \to \infty$, only field configurations that satisfy the flow equations

$$\dot{\phi}^i = \pm g^{ij}\partial_j h \tag{C.2.14}$$

will contribute to correlation functions: instantons are such that $h(\phi(+\infty)) > h(\phi(-\infty))$, while anti-instantons have $h(\phi(+\infty)) < h(\phi(-\infty))$. Note that the second term kills either instantons or anti-instantons as $\lambda \longrightarrow \infty$ as they are suppressed exponentially by

$$\exp\left(-S_\lambda\right) = \exp\left(-\lambda|h(\phi(+\infty)) - h(\phi(-\infty))|\right) = \exp\left(-\lambda\Delta h_{ij}\right),$$

where $\Delta h_{ij} = h(\phi(p_i)) - h(\phi(p_j))$, where $|p_i\rangle, i = 1 \ldots N_{\mu(p)}$ are ground states with Morse index $\mu(p_i)$. Then, as in (C.2.6) with $\lambda = \hbar^{-1}$, we have the matrix elements

$$\langle p_i|d_\lambda|p_j\rangle = n(p_i, p_j)\exp\left(-S_\lambda\right) = n(p_i, p_j)\exp\left(-\lambda\Delta h_{ij}\right), \tag{C.2.15}$$

where $n(p_i, p_j)$ is the signed number of instantons between p_i and p_j. With the normalization $\langle p|p\rangle = 1$, (C.2.15) gives a coboundary operator:

$$\delta|a\rangle = d_\lambda|a\rangle = Q|a\rangle = \sum_{\mu(b)=\mu(a)+1} n(a,b)\exp\left(-\lambda\Delta h_{ab}\right)|b\rangle, \tag{C.2.16}$$

where in the first and second equation we used the identification (??) and (C.2.1) for the supercharge Q and d_λ in the Landau-Ginzburg model. Note that this operator automatically obeys $\tilde{\delta}^2 = 0$, since $d_\lambda^2 = Q^2 = 0$. Moreover we have a selection rule: this matrix element is only non-zero when the degree (in terms of forms) or fermion number between $|p_i\rangle$ and $|p_j\rangle$ differs by 1: the operator d_λ will absorb this unit difference, since d_λ increases the fermion number by 1. Moreover, if p_i has Morse index q and p_j has Morse index $q+1$, then a necessary condition for a flow from p_i to p_j is that $h(\phi_{p_i}) < h(\phi_{p_j})$, since the Morse function is strictly increasing along downward flows. By supersymmetry, there is one bosonic zero-mode, which is the reparametrization invariance of the instanton.

By rescaling states $|i\rangle \longmapsto \exp\left(-\lambda\phi_i\right)|i\rangle$ we obtain our desired result (C.2.12). We see explicitly that supersymmetry transformations tell us about the vacuum structure of supersymmetric quantum mechanics. This picture generalizes to higher dimensions and theories with more supersymmetry, as we will see in chapter 8.

Having established this, we can determine what the true quantum ground states are of the system: they are the ground states associated to critical points that are not connected to critical points that differ by 1 in Morse index. Note that this implies that for a perfect Morse function, there is no instanton tunneling. Moreover, this directly shows that the quantum ground states sit in the cohomology of the coboundary operator δ: a quantum ground state obeys $\delta|p\rangle = 0$ and $|p\rangle \neq \delta|q\rangle$ for some q, since there is no tunneling to $|p\rangle$. Dually, this space is captured by the homology groups of the dual to the Morse-Witten complex.

The deformed 2-sphere revisited

Consider the height function h on the deformed 2-sphere with 4 critical points. Let's label the critical points $|x\rangle, |y\rangle, |z_1\rangle, |z_2\rangle$. Then there are two upward flows from $|x\rangle$ to $|y\rangle$, with opposite orientations. Therefore, $Q|x\rangle = 0$. There is one upward flow from y to z_1 and one from y to z_2: their orientations are opposite too, so we find that $Q|y\rangle = |z_1\rangle - |z_2\rangle$. There is no upward flow from $z_{1,2}$, hence $Q|z_{1,2}\rangle = 0$. Hence, only $|x\rangle$ and a linear combination of $|z_1\rangle$ and $|z_2\rangle$ are quantum ground states.

It turns out that the (co)homology of the Morse-Witten complex is actually independent of the choice of Morse-Smale (h,g) on M. This is quite remarkable, but makes life easier by letting us pick our favorite Morse-Smale pair to calculate with.

C.3 Interpolation between critical points in Picard-Lefschetz theory

In chapter 4 and thereafter we assumed that there was no flow between critical points. Let us briefly discuss when this might happen. We are only using holomorphic polynomials $\mathcal{S}$, with $h = \text{Re}\,\mathcal{S}$. If we assume that we can pick a Kähler metric $g_{i\bar{j}}$ on M, which will be the generic situation for us, then we can write the downward flow equation in local complex coordinates $(z^i, z^{\bar{i}})$:

$$\frac{dz^i}{ds} = -\tilde{g}^{i\bar{j}}\frac{\partial h}{\partial z^{\bar{j}}} = -\frac{1}{2}\tilde{g}^{i\bar{j}}\frac{\partial(\mathcal{S}+\overline{\mathcal{S}})}{\partial z^{\bar{j}}} \Rightarrow \frac{dz^i}{ds} = -g^{i\bar{j}}\frac{\partial\overline{\mathcal{S}}}{\partial z^{\bar{j}}}, \qquad \frac{dz^{\bar{i}}}{ds} = -g^{\bar{i}j}\frac{\partial\mathcal{S}}{\partial z^j}. \tag{C.3.1}$$

Here $\overline{\mathcal{S}}$ denotes the complex conjugate and we denoted by $\tilde{g}_{i\bar{j}} = 2g_{i\bar{j}}$ the metric without a factor of $\frac{1}{2}$ absorbed. From this we find that

$$\frac{d\text{Im}\,\mathcal{S}}{ds} = \frac{1}{2i}\frac{d(\mathcal{S}-\overline{\mathcal{S}})}{ds} = \frac{1}{2i}\left(\frac{\partial\mathcal{S}}{\partial z^i}\frac{dz^i}{ds} - \frac{\partial\overline{\mathcal{S}}}{\partial z^{\bar{i}}}\frac{dz^{\bar{i}}}{ds}\right) = \frac{1}{2i}\left(-\frac{\partial\mathcal{S}}{\partial z^i}g^{i\bar{j}}\frac{\partial\overline{\mathcal{S}}}{\partial z^{\bar{j}}} + \frac{\partial\overline{\mathcal{S}}}{\partial z^{\bar{i}}}g^{\bar{i}j}\frac{\partial\mathcal{S}}{\partial z^j}\right) = 0. \tag{C.3.2}$$

So along a flow, Im $\mathcal{S}$ is conserved. Moreover, we already saw that Re $\mathcal{S}$ strictly decreases along a non-trivial flow. Hence we see that for two critical points p, q to be connected by a nontrivial flow line, we need that Im $\mathcal{S}(p) =$ Im $\mathcal{S}(q)$ and Re $\mathcal{S}(p) \neq$ Re $\mathcal{S}(q)$. We shall mainly use the negative result in our applications: we do not want or need flows between critical points in the hereafter. It is clear that this is not too simplifying: for generic λ, Im $\mathcal{S}$ will always be different at different critical points (in slightly fancier words: we saw already for the Airy function that the Stokes rays form a set of measure zero).

However, it is interesting to explore what might happen when interpolating flows do exist. This is already clear for the Airy function: at the critical points $z = \pm 1$ we have Im $\mathcal{S}(\pm 1) = \mp\frac{2}{3}$Re λ, so when λ is purely imaginary there can be and is a flow line connecting the two critical points, it is just the interval $[-1, 1]$. However, when $\lambda = 0$, $\mathcal{S}$ is trivial and not Morse, so we see that there are two rays, the strictly positive and strictly negative imaginary axis in the complex λ plane for which is an interpolating flow. One calls these rays *Stokes rays*. So we see that there is a nontrivial structure in the complex λ-plane and further analysis of this situation leads to *wall-crossing effects*: as we cross a Stokes ray, the integer coefficients $n_{\pm 1}$ may jump, but the cycles defined by downward flow 'jump' too, as to keep the linear combination $\mathcal{C} = n_{-1}\mathcal{C}_{-1} + n_{+1}\mathcal{C}_{+1}$ constant. We shall not go into detail on this phenomenon here, but refer to [9] for more discussion and references on this. Note that here we can still have interpolation between critical of the same Morse index, as we do not have an obstruction to this due to supersymmetry here.

D

A note on Chern-Simons theory

D.1 Some calculations in Chern-Simons theory

We collect here some calculations in Chern-Simons theory.

Extending the G-bundle on $M \times [0,1]$. Letting $tA, t \in [0,1]$ be a connection on $\tilde{E} \longrightarrow M \times [0,1]$, we can calculate:

$$\begin{aligned}
d(\mathrm{CS}(tA)) &= d\,\mathrm{tr}\left(tA \wedge d(tA) + \frac{2}{3} tA \wedge tA \wedge tA\right) \\
&= \mathrm{tr}(d(tA) \wedge d(tA) + \frac{2}{3} 3t^2 dt \wedge A \wedge A \wedge A \\
&\quad + \frac{2}{3} t^3 \left(dA \wedge A \wedge A + A \wedge dA \wedge A + A \wedge A \wedge dA\right)) \\
&= \mathrm{tr}(d(tA) \wedge d(tA) + 2t^2 dt \wedge A \wedge A \wedge A + 2t^3 \left(dA \wedge A \wedge A\right))
\end{aligned}$$

But we have for the curvature F_{tA} of tA

$$\begin{aligned}
\mathrm{tr}\left(F_{tA} \wedge F_{tA}\right) &= \mathrm{tr}\left((d(tA) + tA \wedge tA) \wedge (d(tA) + tA \wedge tA)\right) \\
&= \mathrm{tr}\left(d(tA) \wedge d(tA) + d(tA) \wedge tA \wedge tA + \wedge tA \wedge tA \wedge d(tA)\right) \\
&= \mathrm{tr}(d(tA) \wedge d(tA) + t^2 \left(dt \wedge A + tdA\right) \wedge A \wedge A \\
&\quad + t^2 A \wedge A \wedge (dt \wedge A + tdA))) \\
&= \mathrm{tr}\left(d(tA) \wedge d(tA) + 2t^2 dt \wedge A \wedge A \wedge A + 2t^3 dA \wedge A \wedge A\right)
\end{aligned}$$

from which we conclude that equation (??) holds.

The winding number. Let us first indicate how to show (7.1.4). Note that for $\mathfrak{g}$-valued 1-forms, we need to be careful in using the cyclic property of the trace: there will be extra minus signs coming from switching around the 1-forms. The rule we need the most is $\mathrm{tr}\,(ABC) = (-1)^2\,\mathrm{tr}\,(CAB) = \mathrm{tr}\,(CAB)$ for A, B, C $\mathfrak{g}$-valued 1-forms. We will leave wedges implicit and repeatedly use $d(g^{-1}) = -g^{-1}dgg^{-1}$. The first term becomes

$$\begin{aligned}
\mathrm{tr}\left(A'dA'\right) &= \mathrm{tr}\left[(gBg^{-1} - dgg^{-1})d(gBg^{-1} - dgg^{-1})\right] \\
&= \mathrm{tr}\left[\left(gBg^{-1} - dgg^{-1}\right)\left(dgBg^{-1} + gdBg^{-1} - gBd(g^{-1}) + dgd(g^{-1})\right)\right] \\
&= \mathrm{tr}[B^2g^{-1}dg + gBdBg^{-1} + B^2g^{-1}dg - b(g^{-1}dg)^2 \\
&\quad - (g^{-1}dg)^2B - dgdBg^{-1} - (g^{-1}dg)^2B + (dgg^{-1})^3] \\
&= \mathrm{tr}\left[BdB + 2B^2(g^{-1}dg) - 3B(g^{-1}dg)^2 - dgdBg^{-1} + (dgg^{-1})^3\right]
\end{aligned}$$

The second term is

$$\begin{aligned}\operatorname{tr} A'^3 = \operatorname{tr}[gBg^{-1} - dgg^{-1})^3] &= \operatorname{tr}[(gBg^{-1} - dgg^{-1})(gB^2g^{-1} - gBg^{-1}dgg^{-1} \\ &\quad - dgBg^{-1} + (dgg^{-1})^2)] \\ &= \operatorname{tr}(B^3 - B^2g^{-1}dg - B^2g^{-1}dg + gBg^{-1}(dgg^{-1})^2 - g^{-1}dgB^2 \\ &\quad + (g^{-1}dg)^2B + (g^{-1}dg)^2B - (dgg^{-1})^3) \\ &= \operatorname{tr}\left[B^3 - 3B^2(g^{-1}dg) + 3(g^{-1}dg)^2B - (dgg^{-1})^3\right]\end{aligned}$$

Hence we get that

$$\begin{aligned}\operatorname{tr}\left(A'dA' + \frac{2}{3}A'^3\right) &= \operatorname{tr}\left(BdB + \frac{2}{3}B^3 - B(g^{-1}dg)^2 - dgdBg^{-1} + \frac{1}{3}(dgg^{-1})^3\right) \\ &= \frac{4\pi}{k}\mathrm{CS}(B) + \operatorname{tr}\left(dgBd(g^{-1}) - dgdBg^{-1}\right) + \frac{1}{3}\operatorname{tr}\left(dgg^{-1}\right)^3 \\ &= \frac{4\pi}{k}\mathrm{CS}(B) - d\operatorname{tr}\left(dgBg^{-1}\right) + \frac{1}{3}\operatorname{tr}\left(dgg^{-1}\right)^3 \\ &= \frac{4\pi}{k}\mathrm{CS}(B) - d\operatorname{tr}\left(-gBg^{-1}dgg^{-1}\right) + \frac{1}{3}\operatorname{tr}\left(dgg^{-1}\right)^3 \\ &= \frac{4\pi}{k}\mathrm{CS}(B) - d\operatorname{tr}\left(gBd(g^{-1})\right) + \frac{1}{3}\operatorname{tr}\left(dgg^{-1}\right)^3 \end{aligned} \tag{D.1.1}$$

which is exactly equation (7.1.4).

Normalization of the trace. Let us now calculate the integral of the third term for G in the special case of gauge theory on S^3, explicitly we want to show (7.1.5)

$$\frac{1}{12\pi}\int_{S^3}\operatorname{tr}\left(dgg^{-1}\right)^3 = \frac{1}{12\pi}\int_{S^3} d^3y\epsilon^{ijk}\operatorname{tr}\left(\partial_i gg^{-1}\partial_j gg^{-1}\partial_k gg^{-1}\right) \in 2\pi\mathbb{Z}. \tag{D.1.2}$$

By a theorem of Bott, any continuous mapping $S^3 \longrightarrow G$ can be continuously deformed into a map $S^3 \longrightarrow SU(2) \subset G$ for simple compact G that have an $SU(2)$-subgroup.

We may parametrize a point $y = (y^0, y^i) \in S^3, i = 1,2,3$ with $(y^0)^2 + (y^i)^2 = 1$, using the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$

$$\sigma_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad \sigma_2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \qquad \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{D.1.3}$$

as generators for $\mathfrak{su}(2)$, we can write a gauge field g as $g(y) = y^0 - iy^k\sigma_k$. We normalize the trace such that $\operatorname{tr}\left(\sigma_i\sigma_j\right) = 2\delta_{ij}$, which amounts to just the standard trace. $g(y)$ is just the identity map $S^3 \longrightarrow SU(2)$, hence the integrand for this map must be a constant. Hence we can evaluate the integrand at a special value of y, so we make the convenient choice $y^0 = 1, y^i = 0$. We can then set:

$$g^{-1} = 1 \qquad \partial_j g = -i\sigma_j \tag{D.1.4}$$

Then the integral reduces to

$$\frac{1}{12\pi}\int_{S^3} d^3y(-i)^3\epsilon^{ijk}\operatorname{tr}\left(\sigma_i\sigma_j\sigma_k\right) \tag{D.1.5}$$

The trace evaluates to

$$\epsilon^{ijk}\operatorname{tr}\left(\sigma_i\sigma_j\sigma_k\right) = \frac{1}{2}\epsilon^{ijk}\operatorname{tr}\left(\left[\sigma_i,\sigma_j\right]\sigma_k\right) = \frac{1}{2}\epsilon^{ijk}\cdot 2i\epsilon_{ijm}\operatorname{tr}\left(\sigma_m\sigma_k\right) \tag{D.1.6}$$

$$= i\epsilon^{ijk}\epsilon_{ijm}2\delta_{mk} = 2i\cdot -6 = -12i \tag{D.1.7}$$

Together with the volume of the 3-sphere $\int_{S^3} d^3y = 2\pi^2$, we find that

$$\frac{1}{12\pi}\int_{S^3} d^3y(-i)^3\epsilon^{ijk}\,\mathrm{tr}\left(\sigma_i\sigma_j\sigma_k\right) = \frac{(-i)^3}{12\pi}\cdot -12i\cdot 2\pi^2 = 2\pi. \tag{D.1.8}$$

The map $g(y)$ maps S^3 once onto $SU(2) \cong S^3$. If we would have taken the nth power of this map, we would have described n coverings of S^3, which would multiply this result of a factor of n. Most notably, this fails for G that do not have an $SU(2)$ subgroup, an example is $SO(3)$. In that case, one obtains the result that the above integral is actually π, exactly half of our calculated result. This reflects the fact that $SU(2)$ is a double cover of $SO(3)$.

The equation of motion. Writing in index notation, we derive (7.1.7)

$$\begin{aligned}
\delta I_{CS} =& \delta\left(\frac{k}{4\pi}\int_M d^3x\epsilon^{ijk}\,\mathrm{tr}\left(A_i\partial_jA_k + \frac{2}{3}A_iA_jA_k\right)\right)\\
=&\frac{k}{4\pi}\int_M d^3x\epsilon^{ijk}\,\mathrm{tr}\left(\delta A_i\partial_jA_k + A_i\partial_j\delta A_k + \frac{2}{3}\left(\delta A_iA_jA_k + A_i\delta A_jA_k + A_iA_j\delta A_k\right)\right)\\
=&\frac{k}{4\pi}\int_M d^3x\epsilon^{ijk}\,\mathrm{tr}\left(\delta A_i\partial_jA_k + \partial_jA_k\delta A_i + \frac{2}{3}\left(\delta A_iA_jA_k + \delta A_jA_kA_i + \delta A_kA_iA_j\right)\right)\\
=&\frac{k}{4\pi}\int_M d^3x\epsilon^{ijk}\,\mathrm{tr}\left(2\delta A_i\partial_jA_k + \frac{2}{3}\left(3\delta A_iA_jA_k\right)\right)\\
=&\frac{k}{4\pi}\int_M d^3x\epsilon^{ijk}\,\mathrm{tr}\left(2\delta A_i\left(\partial_jA_k + A_jA_k\right)\right)
\end{aligned}$$

which gives the desired result. Here we used that the ϵ-tensor obeys $\epsilon^{ijk} = \epsilon^{jki} = \epsilon^{kij}$ and we suppressed the group generators. However, in every step the gauge fields flip positions twice, which is compatible with the cyclic property of the trace to get the generators in the right order.

D.2 A closer look at the Chern-Simons - WZW duality

Wess-Zumino-Witten theory describes maps $g:\Sigma \longrightarrow G$, where Σ bounds a 3-fold B (we can think of B in our case one half of M after slicing it at Σ). The explicit form of the WZW action is

$$S_{WZW}[g] = \frac{k}{4\pi}\int_\Sigma \mathrm{tr}\left((g^{-1}\partial g)(g^{-1}\bar{\partial}g)\right) + \frac{k}{24\pi}\int_B \mathrm{tr}\left(g^{-1}dg\right)^3 \tag{D.2.1}$$

where $k \in \mathbb{Z}$ is an integer for exactly the same reasons in Chern-Simons theory. Here we used that the field g can be extended to a field $g: M \longrightarrow G$, but this extension is not unique: two inequivalent extensions will differ by a constant multiple of 2π in their action. The third term can be written as a total derivative, so it only contributes a 2-dimensional term to the equations of motion. Solving these equations of motion, one finds that there are two conserved currents

$$J(z) = kg^{-1}\bar{\partial}g \qquad J(\bar{z}) = kg^{-1}\partial g. \tag{D.2.2}$$

We can expand the currents as $J(z) = J^a(z)t^a, J(\bar{z}) = J(\bar{z})^a t^a$, which satisfy the operator product expansion

$$J^a(z)J^b(w) = \frac{k\delta^{ab}}{(z-w)^2} + f^{ab}{}_c\frac{J^c(w)}{(z-w)} + \dots. \tag{D.2.3}$$

Using the identity $\partial_{\bar{z}}(z-w)^{-1} = \pi\delta^2(z-w)$, this OPE is alternatively written as

$$\partial_{\bar{z}}J^a(z)J^b(w) = -\frac{k\pi}{2}\delta^{ab}\partial_z\delta^2(z-w) + \pi f^{abc}\delta^2(z-w)J^c(w). \tag{D.2.4}$$

This operator product expansion is equivalent to the Kac-Moody algebra. Consider now coupling the Wess-Zumino-Witten theory to a *background gauge field* $A_{\bar{z}}$ (it has no derivative terms in the action), where for the moment set $A_z = 0$. Then we can consider the partition function

$$Z(A) = \langle \exp\left(-\frac{1}{\pi}\int_\Sigma A^a_{\bar{z}} J^a\right)\rangle. \tag{D.2.5}$$

The background gauge field sources the WZW currents, which means that we can compute correlators of the currents by taking functional derivatives with respect to $A^a_{\bar{z}}$. If we act with the flatness constraint of Chern-Simons theory on $Z(A)$, we find

$$\begin{aligned}\left(\delta^{ac}\partial_{\bar{z}} + f^{abc}A^b_{\bar{z}}(z)\right)\frac{\delta}{\delta A^c_{\bar{z}}(z)} Z(A) &= \frac{k}{2\pi}\partial_z A^a_{\bar{z}}(z) Z(A) \\ -\frac{1}{\pi}\left(\delta^{ac}\partial_{\bar{z}} + f^{abc}A^b_{\bar{z}}(z)\right)\langle J^c \exp\left(-\frac{1}{\pi}\int A^d_{\bar{z}} J^d\right)\rangle &= \frac{k}{2\pi}\partial_z A^a_{\bar{z}}(z)\langle \exp\left(-\frac{1}{\pi}\int A^d_{\bar{z}} J^d\right)\rangle \end{aligned} \tag{D.2.6}$$

Taking now a functional derivative again and setting $A_{\bar{z}} = 0$, we obtain

$$\begin{aligned} &-\pi \left.\frac{\delta}{\delta A^d_{\bar{z}}(w)}\right|_{A_{\bar{z}}=0} \left(\delta^{ac}\partial_{\bar{z}} + f^{abc}A^b_{\bar{z}}(z)\right)\langle J^c(z)\exp\left(-\frac{1}{\pi}\int A^d_{\bar{z}}J^d\right)\rangle \\ &\quad= -\frac{1}{\pi}\partial_{\bar{z}}\langle J^a(z)J^d(w)\exp\left(-\frac{1}{\pi}\int A^d_{\bar{z}}J^d\right)\rangle|_{A_{\bar{z}}=0} + f^{adc}\delta^2(z-w)\langle J^c(z)\exp\left(-\frac{1}{\pi}\int A^d_{\bar{z}}J^d\right)\rangle|_{A_{\bar{z}}=0} \\ &\qquad - \frac{1}{\pi} f^{abc}A^b_{\bar{z}}(z)\langle J^c(z)J^d(w)\exp\left(-\frac{1}{\pi}\int A^d_{\bar{z}}J^d\right)\rangle|_{A_{\bar{z}}=0} \\ &\quad= -\frac{1}{\pi}\partial_{\bar{z}}\langle J^a(z)J^d(w)\rangle + f^{abc}\delta^2(z-w)\langle J^c(z)\rangle, \end{aligned}$$

while the right-hand side becomes

$$\begin{aligned} &\left.\frac{\delta}{\delta A^d_{\bar{z}}(w)}\right|_{A_{\bar{z}}=0} \frac{k}{2\pi}\partial_z A^a_{\bar{z}}(z)\langle \exp\left(-\frac{1}{\pi}\int A^d_{\bar{z}}J^d\right)\rangle \\ &\quad= \frac{k}{2\pi}\partial_z\delta^{ad}\delta^2(z-w)\langle\exp\left(-\frac{1}{\pi}\int A^d_{\bar{z}}J^d\right)\rangle|_{A_{\bar{z}}=0} - \frac{k}{(2\pi)^2}\partial_z A^a_{\bar{z}}(z)\pi\langle J^d(w)\exp\left(-\frac{1}{\pi}\int A^d_{\bar{z}}J^d\right)\rangle|_{A_{\bar{z}}=0} \\ &\quad= \frac{k}{2\pi}\partial_z\delta^{ad}\delta^2(z-w). \end{aligned}$$

Putting it all together we find that

$$\partial_{\bar{z}}\langle J^a(z)J^d(w)\rangle + \pi f^{abc}\delta^2(z-w)\langle J^c(z)\rangle = \frac{k\pi}{2}\partial_z\delta^{ad}\delta^2(z-w). \tag{D.2.7}$$

which is exactly the operator product expansion for the WZW currents (D.2.4). We see that we can identify the Wess-Zumino-Witten wavefunction as

$$\Psi_{WZW}[A_{\bar{z}}] = \langle \exp\left(-\frac{1}{\pi}\int A_{\bar{z}}J\right)\rangle. \tag{D.2.8}$$

Turning on the gauge field A_z, completely analogous considerations hold for the opposite current $\overline{J}$ of the theory. The JJ, $\overline{JJ}$ OPEs are equivalent to the Ward identities for WZW correlation functions, which can be expanded in a basis of *conformal blocks*. One can expand the full WZW partition function in terms of conformal blocks as

$$Z(A_z, A_{\bar{z}}) = \sum_\alpha \Psi_\alpha[A_z]\Psi_\alpha[A_{\bar{z}}], \tag{D.2.9}$$

which are then *uniquely* determined by the Ward identities [21]. We conclude that the Chern-Simons flatness constraint is equivalent to the current OPE of WZW theory.

D.3 Chern-Simons flow equations and instanton equations

We start from equation (8.1.7), the flow equation for Chern-Simons theory. Our goal is to show that it is equivalent to the (anti-)self-duality equations of twisted $\mathcal{N} = 4$ SYM. Writing (8.1.7) out in components, we have

$$\overline{\mathcal{F}} = d(A - i\phi) + (A - i\phi) \wedge (A - i\phi) = F - id\phi - iA \wedge \phi - i\phi \wedge A - \phi \wedge \phi = F - id_A\phi - \phi \wedge \phi,$$

so the flow equation can be written, dropping wedges, as

$$\frac{dA}{ds} + i\frac{d\phi}{ds} = -(\cos\alpha - i\sin\alpha)\left(F - \phi^2 - id_A\phi\right). \tag{D.3.1}$$

Separating real and imaginary parts we have

$$\frac{dA}{ds} = - *_M (\cos\alpha(F - \phi \wedge \phi) - \sin\alpha d_A\phi) \tag{D.3.2}$$

$$\frac{d\phi}{ds} = *_M (\sin\alpha(F - \phi \wedge \phi) + \cos\alpha d_A\phi) \tag{D.3.3}$$

These equations can rewritten by considering them as defined on $I \times M$, where $s \in I$ is the flow time 1-manifold. We extend the metric g on M to a metric g on $I \times M$ by simply adding a component $g_{ss} = 1$, subsequently, we can extend the Hodge star $*_M$ to a 4-dimensional Hodge star $*$ on $I \times M$. We can fix a gauge in which the components A_s, ϕ_s vanish. Taking linear combinations of the flow equations gives

$$\left(F_{(4)} - \phi^2_{(4)}\right)^+ = u(D\phi_{(4)})^+, \qquad \left(F_{(4)} - \phi^2_{(4)}\right)^- = -u^{-1}(D\phi_{(4)})^-, \tag{D.3.4}$$

where the subscript denotes 4-dimensional fields, $D = d_A$ is the 4-dimensional gauge-covariant derivative, the superscripts $\pm$ denote the self-dual and anti-self-dual parts and

$$u = \frac{1 - \cos\alpha}{\sin\alpha} \qquad u^{-1} = \frac{1 + \cos\alpha}{\sin\alpha}. \tag{D.3.5}$$

The moment map extends easily to $\mu_G = d_A * \phi$, since we gauge-fixed $\phi_s = 0$.

Let us now show that (D.3.4) is equivalent to (D.3.2). We write out the above equations in index notation with $i, j, k = 1, 2, 3$ indices tangent to M and $\mu, \nu = 0, 1, 2, 3$ indices tangent to $I \times M$. We can then decompose the 4-dimensional gauge fields

$$A = A_s ds + A_i dx^i = A_\mu dx^\mu, \qquad \phi = \phi_s ds + \phi_i dx^i = \phi_\mu dx^\mu \tag{D.3.6}$$

and consequently put $F_{si} = \frac{dA_i}{ds}$. The covariant derivative is likewise decomposed as $D = d_A = d_{A_s} + d_{A_i} = ds\partial_s + d_{A_i}$. We drop the subscript (4) for convenience. We can first subtract the two equations in (D.3.4), from which we get

$$\begin{aligned}
*(F - \phi^2) &= u(D\phi)^+ + u^{-1}(D\phi)^- \\
*(F_{\mu\nu} - \phi_{\mu\nu}) &= u(D_{[\mu}\phi_{\nu]})^+ + u^{-1}(D_{[\mu}\phi_{\nu]})^- \\
\sin\alpha * (F_{\mu\nu} - \phi_{\mu\nu}) &= (1 - \cos\alpha)(D_{[\mu}\phi_{\nu]})^+ + (1 + \cos\alpha)(D_{[\mu}\phi_{\nu]})^- \\
\sin\alpha * (F_{\mu\nu} - \phi_{\mu\nu}) &= D_{[\mu}\phi_{\nu]} - \cos\alpha * (D_{[\mu}\phi_{\nu]}) \\
\sin\alpha\epsilon_{\mu\nu\rho\sigma}(F^{\rho\sigma} - \phi^{\rho\sigma}) &= D_{[\mu}\phi_{\nu]} - \cos\alpha\epsilon_{\mu\nu\rho\sigma}(D^{[\rho}\phi^{\sigma]})
\end{aligned}$$

Here we repeatedly used the properties of (anti-)self-dual forms and abbreviated $\phi_{[\mu}\phi_{\nu]} = \phi_{\mu\nu}$. In terms of ϵ-symbols, we have that $*_M$ is represented by ϵ_{0ijk}, so putting $\mu = 0$, we read off that the last line then implies that

$$\sin\alpha\epsilon_{0ijk}(F^{jk} - \phi^{jk}) = D_{[0}\phi_{i]} - \cos\alpha\epsilon_{0ijk}(D^{[j}\phi^{k]}) \tag{D.3.7}$$

Using the decomposition, we see that $D_{[0}\phi_{i]} = D_0\phi_i = \partial_0\phi_i$, since $A_0 = 0$, and we see after rearranging we have

$$D_{[0}\phi_{i]} = \sin\alpha\epsilon_{0ijk}(F^{jk} - \phi^{jk}) + \cos\alpha\epsilon_{0ijk}(D^{[j}\phi^{k]}) \tag{D.3.8}$$

$$\partial_0\phi_i = *_M\left(\sin\alpha(F_{(M)} - \phi_{(M)} \wedge \phi_{(M)}) + \cos\alpha d_A\phi_{(M)}\right) \tag{D.3.9}$$

which is equivalent to the second equation in (D.3.2). Note that by $d_A\phi$ we mean the spacial part of the gauge covariant derivative. and that the subscript makes clear that the fields are the components restricted to M. If we add, instead of subtract, the two equations we get the same result. The second flow equation is obtained along similar lines: switching the factors of u and subtracting the two flow equations one obtains

$$\begin{aligned} u^{-1}(F-\phi^2)^+ + u(F-\phi^2)^- &= (D\phi)^+ - (D\phi)^- \\ (1+\cos\alpha)(F-\phi^2)^+ + (1-\cos\alpha)(F-\phi^2)^- &= \sin\alpha * (D\phi) \\ (F-\phi^2) + \cos\alpha * (F-\phi^2) &= \sin\alpha * (D\phi) \\ (F-\phi^2) &= \sin\alpha * (D\phi) - \cos\alpha * (F-\phi^2). \end{aligned}$$

Now $F_{0i} = \partial_0 A_i$, so in index notation, the $0i$-component of the above equation becomes

$$\partial_0 A_i = -\epsilon_{0ijk}\left(\cos\alpha(F^{jk} - \phi^{jk}) - \sin\alpha(D^{[j}\phi^{k]})\right)$$

$$\partial_0 A = - *_M\left(\cos\alpha(F_{(M)} - \phi_{(M)} \wedge \phi_{(M)}) - \sin\alpha d_A\phi_{(M)}\right). \tag{D.3.10}$$

This is exactly the first equation in (D.3.2).

E

Bibliography

In this section we give some pointers to the literature relevant to the material in this thesis. A lot of the basic material on the toy models and mathematical point of view on supersymmetric QFT and gauge theories can be found in [7]. More information on index theorems and more can be found, for instance, in [4]. The canonical reference for conformal field theory, which we used in the appendix, is the Yellow Book [21].

Topological field theory and topological strings were originally introduced by Witten in the 80s, see for instance [54]. He introduced the twisting procedure and the A and B-model. The interpretation of supersymmetry vacua in SQM in terms of Morse theory was another insight by Witten in [8]. Continuing the connection, in the seminal paper [19], Witten showed the connection between Chern-Simons theory and knot invariants. More details on Chern-Simons theory can be found in [55, 51, 50, 18, 56, 57]

The recent application of Morse theory to find exotic integration cycles was developed in the series [12, 29], again pushed forward by Witten. He subsequently published his conjecture on Khovanov homology in [29]. Khovanov originally introduced the categorification of the Jones polynomial in 2000, starting with [23]. An accessible introduction to Khovanov homology can be found in [58].

For symplectic geometry, Gromov-Witten invariants and Floer theory, a good reference is [13].

Background on instantons, solitons can be found in [59]. The basic reference for string theory and M-theory is [2]. More information on the geometric Langlands program and S-duality can be found in [17, 60]. The discovery of coisotropic branes is also due to Kapustin in [6]. A good start for mirror symmetry is [7].

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